THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE THIRD YEARBOOK

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE THIRD YEARBOOK

SELECTED TOPICS IN THE TEACHING OF MATHEMATICS

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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The National Council of Teachers of Mathematics is a national organization of mathematics teachers whose purpose is to

1. Create and maintain interest in the teaching of mathematics.
2. Keep the values of mathematics before the educational world.
3. Help the young and inexperienced teacher to become a good teacher.
4. Improve teachers in service by making them better teachers.
5. Raise the general level of instruction in mathematics.

All persons interested in mathematics and in the teaching of the subject are eligible to membership. Each member receives the official journal of the National Council (*The Mathematics Teacher,* which appears monthly from October to May inclusive) by remitting the annual dues of $2.00. All correspondence relating to editorial matters, subscriptions, advertising, and other business matters should be addressed to *The Mathematics Teacher,* 525 West 120th Street, New York City.

EDITORS' PREFACE

The first two yearbooks of The National Council of Teachers of Mathematics have been so well received that the Council has decided to publish others as long as the interest in these yearbooks continues. The plan is to have each yearbook appear in time for the annual meeting of the National Council in February on the Friday and Saturday just prior to the annual meeting of the Department of Superintendence of the National Education Association.

The first yearbook dealt with "A General Survey of Progress in the Last Twenty-Five Years"; the second was devoted to a consideration of "Curriculum Problems in Teaching Mathematics"; and the third deals with "Selected Topics in the Teaching of Mathematics." We have selected for discussion in this yearbook topics in which teachers of mathematics are interested and concerning which there has been some difference of opinion. Moreover, some of the topics were chosen because we felt that the teaching of them might be improved if we had more available reference material concerning them.

We wish to take this opportunity to express our genuine appreciation to all who have helped in the preparation of this yearbook, especially those who contributed the discussions in the various chapters.

THE EDITORS

CONTENTS

SELECTED TOPICS

IN THE

TEACHING OF MATHEMATICS

THE FALLACY OF TREATING SCHOOL SUBJECTS AS "TOOL SUBJECTS" *

By CHARLES H. JUDD

School of Education, University of Chicago

The Significance of Words. The psychologist who has studied the evolution of language is perhaps more impressed than are others by the fact that words embody and transmit profound beliefs. It has been said that the phrases "struggle for existence" and "survival of the fittest" have done more to direct modern thinking than whole volumes of statutes passed by parliament or congress. We all know the limits to which industrial and commercial concerns will go in order to secure selling names for their wares. It is a common observation that parents in conferring on their offspring the designations which are to mark them throughout life seek names which express the hope that greatness of one form or another will crown the individual's career.

I do not know who first called arithmetic and reading "tool subjects." Whoever he was, he stigmatized these subjects and classified them as in some sense inferior to history and science and literature. Suppose that we could have been present at the first use of the term "tool subjects" and could have persuaded the company not to use that term but to call reading and arithmetic the "right and left hands of learning," what would have been the effect on subsequent educational thinking of the adoption of our more complimentary terms?

I am not suggesting an idle speculation about the history of words. My psychology tells me that words are the embodiments of discriminations and the epitomes of theory. My psychology teaches me that when the zoölogist points to an animal and says: "That is a vertebrate," he has condensed into a single word the study of generations of scientists and has pointed out the characteristic of the animal which long study has shown to be the essential fact on which attention is to be concentrated if we are to understand the place of that animal in the organic world.

* Paper presented to the National Council of Education at Dallas, February, 1927.

Arithmetic as a "Tool Subject." When a writer on education says arithmetic is a "tool subject," what does he mean? We have only to appeal to present-day educational literature to secure a clear answer. Arithmetic is thought of as a subject which is a kind of necessary evil. If one wants to pry loose interest from capital, one will have to do a little calculating. If one wants to purchase bricks and lumber for the highly important business of building a house, one will have to delay in the truly constructive work long enough to add up a column of figures. If one is to engage in the interesting occupation of compounding a recipe, one will have to stop and measure and count the ingredients. The ends of life are possessions, and buildings, and food; these are the matters with which many educators declare themselves to be concerned; these, they tell us, are the substantial sources of valuable mental experience; these are the concrete facts of life. Number is to the minds of many of our contemporaries a formal abstract something which has crept into the schools and crowded out property, and buildings, and food. Ardent critics of mathematics would have us believe that number is a tool which one uses perhaps twice a day, possibly three times, but never without apologies and never without a feeling of aversion at the delay which number thinking entails. The popular plea today is a plea for reduction all along the line. Let us find out how frequently the people of Iowa and Boston really use arithmetic in their daily lives, let us be rid of antique puzzles about ditch digging and the rest, let us laugh algebra and geometry out of the high school.

Present Emphasis on the Concrete. As I read the modern textbook in arithmetic I find that the tables are few and short. There is concreteness on every hand. There are pictures of people buying and selling, there are reproductions of checks and bills of lading, there are problems from geography and baseball, but the fact that the number series has a character and regularity of its own is pushed as far into the background as the pages will permit. The new textbooks might have on their covers a picture of a tool as symbolic of their content. They seem to announce in the loudest tones the belief that all the good things in modern civilization are real, tangible objects; number is merely something to be used now and then as a vague and unsatisfactory substitute for things that are concrete and substantial and truly important. In other words, they seem to think that it is a necessary evil.

Why the Attack on Arithmetic? I have sometimes wondered how far this attack on arithmetic is what the students of mental pathology call a "defense reaction." When a psychiatrist finds his patient overviolent in his protestations along any line, he begins to look for the cause of the mental derangement in the direction indicated. Here is a sensitive soul who tries to protect himself from the crushing effects of social criticism by saying that he does not care what people say about him. Here is a man who has made a failure; we find him boasting to such a degree that the psychiatrist knows that a failure-complex is the source of his functional disturbance.

Whether I am right or not in thinking that the pedagogical attack on arithmetic is due to a failure-complex will, of course, have to be judged in the light of educational history which reaches beyond our own day. Certain it is that the schools of our own time do not know how to teach arithmetic. The net result of the testing which we have done since the first paper on arithmetic tests was published by J. M. Rice in 1902 is a shocking revelation of the incompetency of the schools in teaching arithmetic. The Cleveland survey showed a decade ago a fact which is revealed again in the California Curriculum Study which has recently come from the press, namely, that more pupils fail in arithmetic in the third grade than in the second, more fail in arithmetic in the fourth grade than in the third. From the fifth grade on there seems to be a falling off in the number of failures in arithmetic, but probably the apparent increase in success of the school's operations is due to the elimination of the pupils whom the school can not train. The successive increases in the number of failures are convincing testimony to the difficulty of teaching arithmetic. One suspects, as I said before, that the attitude of teachers toward this subject which they call a mere "tool subject" is in large measure attributable to the fact that they do not know how to teach it successfully.

The Important Uses of Number. I can hardly expect to stem the tide of common opinion by anything that I can say in a single paper, but I am here to urge that the term "tool subject" be carefully reconsidered. For my own part, I reject it absolutely. The experiences which have come into modern life from the study of number are not the trivial rules of addition and subtraction and the rest; they are experiences of a wholly different order. The curriculum maker who thinks that he has exhausted the catalogue

of uses of number when he has listed the examples which ordinary men solve in a day or a week is superficial to such an extreme degree that he is an unsafe guide in arranging the plans of the school. The man who calls arithmetic a "tool subject" and with this name dismisses it as something less worthy than subject-matter courses is guilty of criminal neglect of true values.

Historical Facts. In support of the general statements which I have made, I call attention to certain significant historical facts. Our present-day civilization is one which depends on the universal recognition of the importance of the idea of precision. Every machine which serves us in daily life is precise in its construction; every engagement of life is precise in its temporal arrangement; every economic relation, if it is to be just and acceptable to all concerned, must be precise. Precision is sought in standardization, in the discharge of one's trusts, in the statements which men make to their fellows. Precision is the soul of science and of commerce.

As contrasted with civilization, primitive life is characterized by its lack of precision. Savages do not know how to be exact; they have habits of thought and action which the modern man must regard as absolutely intolerable in their loose and vague character.

The Curriculum Maker's Method. When the curriculum makers ask pupils to collect examples of the use of number, do they tell them to bring in all the cases of explicit or implicit use of the idea of precision? Not at all. The curriculum maker's method is an easy-going method of analyzing life. He looks only at that which is on the surface. Especially when he has adopted the classification of arithmetic as a "tool subject," he is satisfied to think of number as something to be taken up or laid aside as occasion demands and he usually thinks that number is involved only when he can see it openly and explicitly used.

A Plea for Deeper Analysis in Curriculum Making. My plea is for an analysis of mental life which goes much deeper. The pupil who is drilled day after day in the use of number is acquiring a mode of thought which will change all of his later mental operations. The individual who has had experience with number is no longer capable of returning to the level of loose, inexact thinking that characterized his earlier methods of viewing the world. The curriculum maker who would be true to the facts must inquire into the subtle modes of thinking which lie deeper than the mere expressions of number in the ordinary meaning of that word.

The Evolution of Numbers. The fundamental idea of precision is not something that comes easily to men. It is not an instinct nor is it related to an instinct. It is one of the highest products of intellection. It did not become a possession of the common people until civilization had made possible certain forms of coöperative thinking. There are primitive tribes today which have only the most meager number terminology. The peoples of antiquity had the beginnings of a number system but their number ideas were crude. The symbols that they used covered up many of the most important properties of number. For example, the Roman numerals are of little use in addition and they obstruct almost completely the effort to multiply.

Clear number ideas evolved slowly. They had to wait for full development until there had been invented that most important of intellectual systems, the Arabic number system. When the Arabic numerals came into Europe in the sixteenth century, there was no machinery, there was no exact science, there was very little commerce. Men could not think in large terms because they had no intellectual methods for compassing large ideas. Within the short space of less than four hundred years Europe has become the home of a mechanical civilization absolutely dependent for its erection on the general idea of precision. This idea of precision is not a collection of number names, nor a series of addition combinations, nor the combinations taught in subtraction; it is a general idea refined out of many uses of number and ultimately taking its place in the minds of men as a guiding principle dominating every train of thought that a modern man follows.

Number an Ever-guiding Principle of Life. The figure of speech which one ought to use, if figure of speech is necessary, is not the figure of a tool, that is, of something that is now taken in the hand, now laid aside; the figure of speech which is appropriate is one which conveys the idea that number is an ever-guiding principle of life. One assimilates food and grows into a being full of strength and maturity. One lives in the sunlight and his vision is filled with new images and his mind is stimulated to new action. The number system is such an energizing principle. It has changed the life of men. It has become a mode of thinking. It pervades every observation which modern man makes. It can no more be laid aside than can the right hand. It is not a tool.

What has been said about number can be repeated in exactly the

same terms with regard to reading. The pupil who has learned to read will never again lead the kind of intellectual life which he did before he acquired the art. Words will come in a new form into the mind of the pupil who has learned to read. The world of books will be a familiar world and written expressions will be an intimate part of the pupil's inner experience.

Correct Notions about School Subjects. When the organic world evolves a new organ, such as a prehensile extremity or an air-breathing apparatus, no one ever thinks of calling the hand or the lungs tools. No one who has studied zoölogy fails to recognize that an animal with a hand or with lungs enters on a new type of life and becomes by virtue of these new organs a totally different kind of animal from those which preceded it in the evolutionary series.

What I am trying to say cannot be reconciled with that false educational psychology which is current in certain quarters where arithmetic is described as a mastery of so and so many different number combinations and a collection of so and so many distinct associations. Arithmetic is a general mode of thinking. Through long eons of human endeavor men have been moving forward step by step in the world of ideas, discovering now one now another of the regularities which nature exhibits. In the effort to comprehend these discoveries and to arrange them into a system of experience men have grown to the point where their minds have evolved certain general ideas and certain general modes of thought. Number is one of the latest of these evolved modes of thought. It could not be perfected until a system of symbols was invented which could express adequately the ideas of precision and regularity.

Precision, Order, and Arrangement. Along the road toward the complete recognition of these ideas of precision, of order, and of arrangement are distributed bits of evidence showing how the idea was first formulated in detail and gradually generalized. The Roman army was organized into companies of one hundred and the officer in charge of each company was called a centurion, the Latin for one hundred being the word *centum*. A Roman mile was one thousand paces, the Latin for one thousand being *mille*. The names of the months of September, October, November, and December are derived directly from the Latin numerals *septem*, *octo*, *novem*, and *decem*.

These examples show how the Romans were trying to arrange

their world. In the effort to bring order out of the seeming chaos of reality they employed a kind of universal formula. All things can be systematically arranged if only one has within one's self the idea of order and sequence. One can not have the idea of order, however, until one has symbols and some kind of an established sequence of thought.

I confess I am astonished when I see some of the analyses which purport to be the scientific foundations on which school curriculums are to be built and find no mention of these general ideas of order and arrangement and precision. I am told that the school should teach children how to make change and how to measure wall paper and how to tell time and that sections of arithmetic should be devoted to these specific tasks, but I look in vain for any appreciation of the fact that the school ought to lead pupils who have only a hazy and unsystematic notion of the world to see the value of arrangement and order in all thinking and to cultivate the general ideas of regularity and precision.

I venture the prophecy that we are just at the point where we are about to leave behind the inadequate psychology which has in recent years taught that mental life is a bundle of particular ideas. We shall hear more and more in the years to come about general ideas. We shall realize that general ideas do not arise in the untrained mind but are the highest products of constructive thinking. The number system, the mathematical formulas of algebra and geometry, however inefficiently taught in the past, have helped the race to organize and arrange the world in which we live. He who thinks of the number scheme as a trivial addendum to the mind's equipment does not know his history.

The Place of Mathematics in the School Curriculum. I am anxious to array myself as emphatically as I can in opposition to those who argue for an elimination of mathematics from the school curriculum. I am quite prepared to join in a criticism of much of the teaching which goes on in the schools at the present time. Instead of being discouraged by the formalism of present-day courses, I am encouraged by the agitation which has arisen against formalism. I am encouraged even though I see some of the criticisms of formalism turned into attacks on the subject itself. Personally I see no reason for trying to escape the difficulties which arithmetic and algebra and geometry encounter because of inefficient teaching and inadequate understanding of their purpose. These difficulties

should not be evaded by casting the mathematical sciences out of the curriculum. The mathematical sciences have demonstrated their place in the intellectual world. If we do not teach them successfully, let us ask ourselves what are the reasons for our failure and let us ask ourselves whether we have been intelligent in defining the true content of these sciences.

Attitude toward the Mathematical Sciences. Thus far my argument has been directed against the acceptance of the view that number is a remote and external addition to mental life. I turn now to a reinforcement of my contentions by expounding briefly what I think should be the attitude of teachers toward the mathematical sciences.

Pervading all of one's thinking about the world should be the general attitude that the objects which come into experience can be arranged and classified. Order achieved by human effort as a substitute for the world's natural disorder is the end and aim of human intelligent reaction on the world. We are to put things in series, where like objects are together and unlike are distinguished. Order, arrangement, classification are products of the mind's reaction on the world. Whatever else the school does, it should help individuals to systematize their experiences.

Orderly arrangement is by no means easy to produce. One must discriminate and assemble and reassemble. One must put together in the mind what has not been together before. Dr. Osler once made the sage remark that medicine will become a science when physicians learn to count. He meant exactly what I have been saying in less epigrammatic form. When physicians learn to distinguish diseases and count consequences and enumerate methods of treatment and measure results, medicine will be a science.

Little children come into the school at a period of life when they have attained very little in the way of orderly arrangement of experience. They have no notion of regularities and system. It is the business of the school to transmit to the pupils the intellectual methods of arrangement by which the complexities of the world may be unraveled and a new pattern made of experience. The most comprehensive and flexible patterns for the rearrangement of experiences are those supplied by the mathematical sciences.

Let me offer an illustration of what I mean by referring to an analysis which I recently made of a number of standard arithme-

tics. I noted that there is great variety exhibited in the situations described in the problems. For example, some of the problems discuss the finding of things: John had five marbles and found three. Other examples tell about earning money: John earned ten cents doing this and fifteen doing that. Other examples tell about journeys in which a man walked two miles and then four. You can readily supply other similar types of examples. The interesting point in regard to all these cases, whether they be of finding or earning or journeying, is that they belong together under the inclusive notion of addition. I found that the plus sign taken as the symbol of the general idea of addition has in four standard arithmetics somewhat more than four hundred synonyms. There are as many synonyms of the minus sign. Thus John lost three marbles; John paid fifteen cents out of his earnings; the man has thirty miles to go but covered in one day only eighteen.

It is further important for our purpose to note that the general ideas of addition and subtraction stand in a certain intellectual relation to each other that is not indicated on the surface of the problems. John earns or spends; the two performances carry John in opposite directions and the degree in which he moves in one direction or the other can be exactly compared through the use of arithmetical symbols. Thus we see that there is a higher form of mathematical generalization which includes the two contrasting operations of addition and subtraction.

We have all observed much teaching in the schools which consisted in solving problems of earning and spending but which left no clear notion in the pupil's mind of the kind which I referred to above when I spoke of the general ideas of addition and subtraction. How often have we all heard children in the schools say, "I don't know whether to add or subtract, whether to add or multiply." Even more common is the failure to prepare for the higher synthesis of addition and subtraction which is made in algebra.

When we note the failure of ordinary instruction in arithmetic to cultivate general ideas, we can not wonder that so many pupils fail. Teachers have not regarded arithmetic as a science made up of general modes of thinking; they have thought of it as a series of rules. They have called it a "tool subject." They have pictured it as something that the child can lay aside without impairing his view of the world. They have hurried over the tables which show

how orderly and systematic number ideas are because somebody has told them that the mind deals only with particular experiences and that all knowledge is concrete. They have been told that Iowa families get on with very little use of the rules of arithmetic. They have been warned against assuming that an idea gained in one connection can ever be used in any dissimilar surroundings.

The New Doctrine. In place of all these vagaries, it is the purpose of this paper to advocate a doctrine of general ideas and all-pervasive modes of thought. He who is master of the number system has a way of thinking that the race has worked out with infinite labor. He will never again fall back into the confused and inexact ways of viewing the world which were characteristic of his childhood. He has grown intellectually by assimilating the number system. He has a mature mind. He has strengthened his mind. He is a new individual.

If I have persuaded you in any degree, you should join me in criticizing the superficial studies of the curriculum which are trying to limit the study of mathematics to those problems in the solution of which people use some rule of combination. You should join me in resisting the tendency to banish mathematics from the programs of students in the high school. You should join me in protesting against the classification of arithmetic as a "tool subject."

MATHEMATICS IN THE TRAINING FOR CITIZENSHIP *

By DAVID EUGENE SMITH
Teachers College, Columbia University

Purpose. It is the purpose of this paper to set forth certain facts to be kept in mind in the teaching of mathematics for citizenship. It is proposed to consider in the limited space available what there is in mathematics that the working man needs and what of this science the woman requires in directing the education of her children and in managing her home; how mathematics trains the mind and how its poetry affects human life; what potency the subject has in the uplift of your soul and mine; how it has linked itself with all humanity in all times; and how we should go about to make all this real in our everyday teaching.

Thus stated, the problem falls of its own weight,—for in the brief space allowed, it is impossible to treat with any adequacy these various topics and the many questions that naturally arise in your mind and my own. All that can be done is to mention a few of the leading principles that have guided the writer in his own endeavors to improve the teaching of mathematics, and which he feels sure guide a large proportion of his former students in their work in various parts of the world.

First Objective. First, we study mathematics because it is one of the small group of subjects—like reading, history, and geography —that are linked up with a large number of the branches of human knowledge. No one can be happy as a member of the human family who does not know something of the history of the race, something of the earth on which this race exists, something of letters, something of the arts, and something of what we so pedantically call "the quantitative side of human life." Of the necessity for knowing number relations there can be no question, but fifty years ago one might well have cried the slogan abroad from the housetops, "Will anyone tell me why a girl should study algebra?"

* Adapted from an address delivered before the faculty of Teachers College. The original address was published in the *Teachers College Record*, Vol. XVIII, No. 3.

Today, a person would sadly feel his ignorance, or her ignorance, if he or she had to look with lack-luster eyes upon a simple formula such as may be found in *Popular Mechanics*, *Motor*, the *Scientific American*, an everyday article on the radio or astronomy, a boy's manual on the airplane, or any one of hundreds of articles in our popular encyclopedias. These needs come not only within the purview of the boy; they are even more apparent in the case of the girl, she who is to have the direction of the education of the generation next to come upon the stage of action. Each must know the shorthand of the formula, and the meaning of a simple graph, of a simple equation, and of a negative number, or else must feel the stigma of ignorance of the common things that the educated world talks about and reads about.

Reach of Mathematics. And if you are skeptical as to the reach of mathematics in the world in which we live, consider this simple suggestion. Aside from the propagation of the race, the most important thing in this world is education—which term is taken to include the training of the soul as well as the training of the hand and brain, and the training for the eight hours or more of daily leisure as well as for the eight hours or less of toil. Let us then take the science of education as a norm for measurement. Let us now imagine, if we can, that by some mighty cataclysm there should be wiped off the face of the earth tonight every book on mathematics, every mathematical symbol of any kind, every written page or printed sheet upon which a trace of mathematics appears, and every machine for computing or recording numbers; and then let us do the same for every piece of printing or writing that has to do with the science of education. What would happen? "It is an ill wind that blows no good," and some good would unquestionably come to the world were this done. For one thing, all wars and rumors of wars would stop tomorrow, since shells of the right size could not be sent to the proper guns and the range finders would cease to operate. For another thing, we should doubtless have more attention given to real teaching because there would be some lessening of experiment, valuable though this may be in a fair percentage of cases, particularly those in which the approximate measure of pupils' abilities is concerned, and exceedingly valuable as we might make it if we would. The actual teaching in the schools would go along about as usual; very likely, however, with a little less friction for a time. But how about life

beyond our scholastic walls? Every mill in the whole world would slow down and every large concern would close until it could replace its accounts, its statistical material, its formulas for work, its measures, its tables, and its computing machinery. Every ship on the seven seas would be stricken with blindness and would wallow helpless, awaiting the probable starvation of its human burden. Not a rivet would be driven in a skyscraper in New York City, because the steel girders would have lost their numbers; Wall Street would close its portals; the engineering world would awaken tomorrow morning to a living death; the mines would shut down; and trade would relapse to the condition of barter as in the days of savagery. It is a picture that is so ridiculous that we smile at its very impossibility. But it is a real picture, ridiculous though it seems,—a picture of the world sending forth the SOS for help, a hurry call for the return of the mathematics that many take such pleasure in condemning.

But it will at once be said that no one denies the practical value of mathematics, that it is merely a question as to whether a knowledge of our present mathematics is a *sine qua non* to good citizenship; and so to this question I return. A subject even so essential as this in our world economy today need not be mastered by every citizen, and surely no one would think of making any such assertion. What seems reasonable, however, is this: that every educated man or woman should know what mathematics means, what its greatest uses are, and something of its soul, and should thus be able to decide whether or not he or she cares to pursue its study beyond the point of acquiring this elementary knowledge. Upon this I believe that all persons of fair education are agreed, and the only question is how this introduction shall be given and how far it shall lead.

So much for the first point—that everyone should know and must know what mathematics means, at least for the reason that the world uses it so extensively and that everyone comes in contact with such concepts as algebraic formulas, graphs, and simple mensuration.

The Question of Discipline. A second reason for believing that mathematics should enter into the making of a citizen is that it has high value as a mental discipline. It is needless to say that this statement will cause apprehension on the part of many teachers and that it may cause a feeling of exultation on the part of those

who care little for the older disciplines; but the statement is made frankly, and certainly without fear. The validity of the doctrine of mental discipline depends largely upon the definition of the term. If by mental discipline we mean the theory that memorizing the rules of arithmetic has any potent influence on our memorizing the Book of Job; if it means that ability in algebra produces facility in calculation; or if it means that a high mark in digging out the roots of numerical equations correlates with a high mark in determining whether to buy this kind of motor car or that,—then no thoughtful student of education has ever made such an assertion, and so there is no doctrine of formal discipline to combat. The time and energy that have been squandered in proving statistically the axiomatic is one of the interesting phases of modern education.

If, however, the knowledge of how to arrange a logical proof in geometry can be made of no value to us in other fields in which deductive logic can be applied; if the perfection of geometry does not give us an ideal of perfection that helps us elsewhere in our intellectual life; if the succinctness of statement of a geometric proof does not set a norm for statements in non-mathematical lines; if the contact with absolute truth does not have its influence upon the souls of us; if the very style of reasoning does not transfer so as to help the jurist, the physician, the salesman, the publicist, and the educator; if the habit of rigorous thinking, which usually is first begun in demonstrative geometry, is not a valuable habit elsewhere; if a love for beauty cannot be cultivated in geometry so as to carry over to stimulate a love for beauty in architecture,—then let us drop demonstrative geometry from our required courses. From the standpoint of actual measurement, aside from the transferable power it gives us for independent investigation, geometry is not, in that case, worth the time and energy it takes.

But it may be asked if we have any proof that geometry has, or can have, this power, and the answer is, "Certainly,—world experience." Of course such a reply will bring at once the assertion that the one making it has not studied the results of psychological experiments. On the contrary, the assertion that there is no transfer of training from geometry to other lines of activity is not so emphatically made at present as it was a dozen years ago. It is by no means the advanced educator today who denies a disciplinary power to geometry; it is either the educator who is slipping behind in the race, or the one who has never been in the race

at all. If anyone says that we have statistics to show that young people spend a year on a subject whose chief purpose is the logical proving of statements and are not thereby made more logical in other lines of mental activity, now or in the years to come, the reply may properly be that psychologists of high standing and scientific men generally do not believe it; and if he shows us his tables, carried out to the usual $\frac{1}{100}$ of 1 per cent, then may we quite safely say to him that there are just as elaborate tables seeking to prove things in education, which tables have secured degrees for their makers, and yet which are so ridiculous that this writer, at least, is ashamed to have his European friends see them lest they judge all American scholars by such unscientific work.

The Magic of the Mind. A third reason for believing in an introduction to the oldest of all the sciences, for the training of a citizen, is that every man and every woman is entitled to experience what Byron called "the power of thought, the magic of the mind." Our schools do much for the gray practicalities of life, and they are to be commended therefor; they do much for idealism, for the poetic side of our nature, and for all this we should be thankful. Now let us see to it that the poetic side of mathematics is recognized as well as the practical side; let us see to it that we show the world how to use its leisure as well as how to turn the restless wheels of industry. Voltaire remarked that "one merit of poetry few persons will deny; it says more in fewer words than prose"; and why may we not with perfect truth continue,—"one merit of mathematics no one can deny,—it says more in fewer words than any other science in the world"? If "poetry is unfallen speech," then geometry is undebased thought, and if "architecture is frozen music" then it is also the crystallized science of form. Do you smile at the suggestion that mathematics is poetry? So men may smile at the idea that the Book of Genesis is one of the grandest poems of all literature,—but only if they have not read it in the noble language in which it was written in the days when civilization was young. So men may smile at the suggestion that mathematics is an epic of ideas, but only because they have not learned to read it in its own tongue. "All men are poets at heart" —but most of us are like bashful children to confess it.

> "Poets are all who love,—who feel great truths
> And tell them,—"

and what is the *real* teacher of mere algebra but this, or the one

who opens the door to any one of the branches of the science mathematical? The beauty of symmetry,—where else is it found as completely as in mathematics, and where does rhythm play as great a part as here? Why was music looked upon, until recently, as a part of mathematics, if not for the common elements of the two? Why does Nature so often arrange the leaves of plants in accordance with a Fibonacci series, and why does the snow crystal recognize the poetry of the complex sixth roots of unity? You do not recall being taught all this? Then the argument is against the pedagogue, still so often a slave as in ancient times; it is not against the poetry of mathematics. Of course, such an idea will not appeal to the educational iconoclast and may be characterized as mere food for the newspaper paragrapher. The former may say that only a genius in teaching can bring out such values, and the latter may have his little pleasantry about some long-haired, impractical, impecunious professor who says that a table of logarithms is a series of sonnets; but neither need concern us particularly, because the rhythm of mathematics is merely a commonplace truth which teachers of mathematics generally recognize and of which most students are aware, whether they confess it or not.

Eternal Verities. The fourth reason to be suggested is less tangible,—we teach mathematics for citizenship because it is one of the eternal verities, just as we might lead the youth to a great desert or to the mountains, there to commune with his soul; to a solemn tropical forest, "God's first temple," there to feel the uplift of loneliness; to the black silence that surrounds the tomb of Akbar the Great, or to the mists in the cryptomerias that stand guard beside the resting place of Ieyasu. We should lead him to these places because they rouse his soul to a contemplation of the truths that endure. Did you ever think how you might proceed to make an attempt to communicate with Mars by signals? How you would place enormous searchlights to form some picture on the plains of Sahara, some picture that the Martian might be certain is not a mere accidental arrangement, or accidental series of flashes, and yet something that he would understand? It could not be a representation of a living thing, for if Mars has life its forms are surely not shaped like those of Earth. It could not be words in letters which have been transmitted to us through the ages, nor could it be the numerals which had their visible source ages ago in the royal decrees of India. No, it would be none of these; the most

hopeful symbol we could give to attract the attention of a world much older than our own, and probably more refined, would be the figure of the theorem of Pythagoras, the squares on the three sides of a right triangle. The reader may smile at the idea,—but he cannot think of a better symbol; and the reason is that here is one of the verities of the universe. Before Mars was, or the Earth, or the sun, and long after each has ceased to exist, there and here and in the most remote regions of stellar space of the type we know,—the square on the hypotenuse was, and is, and ever shall be equivalent to the sum of the squares on the sides. All our little theories of life, all our childish speculations as to death, all our trivial bickerings of the schools,—all these are but vanishing motes in the sunbeam compared with the double eternity, past and future, of such a truth as this.

Man in the Cosmos. A fifth reason for advocating the study of mathematics in the training of the citizen is that it makes him conscious, as is possible in no other way, of his position in the universe about him. To come to some realization not merely of the depths of known space but of the methods by which we sound the plummet of these depths, to know the connection of the mathematical spiral with the genesis of the solar systems, to know something of the way in which we measure the motions of the heavenly bodies, to have some conception of the methods by which we weigh the stars, to know how we reckon time in "light years" and the basis of our assertion that the light which left a certain star when Homer was reciting the fall of Troy is only just reaching us,—think of the glories which then our universe assumes in the mind of the impressionable youth! "What is man that Thou art mindful of him?" "A thousand years in Thy sight are but as yesterday when it is past, and as a watch in the night." And to these words we may add those attributed to the greatest of all philosophers, "God eternally geometrizes."

Religious Bearing. And this suggests a sixth reason which to this writer is very strong—that the proper study of mathematics gives humanity a religious sense that cannot be fully developed without it. We may take all the selections of little groups that we wish, we may show and show again that the mathematician does or does not go to church, that he does or does not believe in priestcraft, that he has or has not faith in this kind of God or that, —all this has no influence upon this thesis, and should have none

because it has no bearing whatever upon the question. In the history of the world, mathematics had its genesis in the yearning of the human soul to solve the mystery of the universe in which it is a mere atom. Mathematics did not come into being as an aid to finance, because finance developed only a few seconds ago on the face of the clock of time; it seems not to have come into being as an aid to measuring land or granaries, nor, save in the simplest counting on the fingers, for any utilitarian reason whatever. It seems rather to have had its genesis as a science in the minds of those who followed the courses of the stars, to have had its early applications in relation to religious formalism, and to have had its first real development in the effort to grasp the Infinite. And even today, even after we have been pushing back the sable curtains for so many long centuries,—even today it is the search into the Infinite that leads us on. Consider the familiar illustration of the points within a sphere, say of radius 1 inch. For a point $\frac{1}{2}$ inch from the center, on any given line, there is one and only one point in space such that the product of the distances of these two points from the center is equal to 1, and this is the point on the line 2 inches from the center. Likewise to a point 10 inches from the center, on any line, there corresponds a point $\frac{1}{10}$ of an inch from the center. And in general, for any point we may select within this little sphere, there is one point and only one in the universe to correspond to it. Conversely, for any point in the universe, of which the galaxy which we call in our vernacular the milky way is a small part, there is one point and only one that corresponds to it within the sphere which one may hold in his hand. There is no point in the whole universe that has not one corresponding point and only one in some brain cell of each of us. In all seriousness we may say: "Behold, I show you a mystery"; and again, "The Kingdom of God is within you."

Human History. And finally, let us consider one more reason why an introduction to mathematics is worthy the attention of the future citizen, namely, that the history of mathematics is the history of the race. From the days when the world was emerging from savagery and counted slowly on its fingers, from the days of wonder of the childlike race as to the mystery of the numbers 3 and 7, from the period when the rope-stretchers planned the altars of India and laid out the temples of Babylon and of Thebes the Magnificent, from the years when Pythagoras founded the world's first

notable university and sought the relation of the geometric solids to the philosophy of life, from the time when Plato set the first entrance examination over the portals of the grove of Academos—"Let no one ignorant of geometry enter here," from the founding of Alexandria and of Bagdad, with their great schools of mathematics—all down through the ages, the history of mathematics has been the history of the race.

Are We Successful? But do we recognize all this in our teaching of the subject? Alas! we recognize so little in our teaching of anything. We fail to recognize adequately the story of the development of the rights of the individual in our teaching of history; we commonly fail to recognize the noble chants of Gregory in the teaching of music; we fail to recognize the sonorous language of the Athenian and the real significance of the oration on the crown in our teaching of Greek; and we fail to grasp the overwhelming significance of education when we spend our weary hours in reports, and in questionnaires, and in regulating the bells of our classrooms.

Frankly, we cannot sanction, as perfect, anything that is in this poor, storm-tossed world of ours, and so we cannot fully sanction the present teaching of mathematics or of anything else. Surely our governments are largely failures; and surely our Christianity, our Judaism, our Mohammedanism, our Buddhism,—these failed to ward off the great blow of savagery that staggered the world in the fateful years of 1914-1918. Our divorce courts, our children's courts, our brothels, our crime, our poverty—all these cry to heaven of the failure of our social system. There exists nothing in this world that cannot be made better, and so let it be said at once that mathematics is poorly taught, but no more poorly than pedagogy; that it contains a mass of material of questionable value to young people, but no more than sociology; that its subject matter can undoubtedly be much better arranged, but no better than can that of psychology; that its beauties are not made adequately manifest, but the same can be said of music and belles-lettres. No one has any brief for the perfection of the teaching of mathematics as it stands; but if we had to take a brief for this subject, or for pedagogy, or for practical arts, or for courses in social betterment, we might with perfect confidence take the mathematics, and we would be justified in so doing because it has a more scientific, logical basis, and because it has had a far longer time in which its method could develop toward perfection.

When we think of the record of failure in the proper presentation of such subjects as religion, and art, and pedagogy, and mathematics, we may sometimes lose heart,—but only for a moment, for we know that we are constantly tending toward the good. Someone may tell us that not 0.1 per cent of pupils attain a mark of 100 in algebra, and that 30 per cent fail altogether; but when we reflect that not 0.001 per cent of humanity attains a mark of perfect in matrimony, and that many more than 30 per cent fail altogether, we do not at once begin a crusade to abolish marriage. Nor is this analogy either far-fetched or ridiculous. If anyone says that we must have matrimony or else the race will perish, a fair reply is that this statement is not scientifically accurate, nor has it been scientifically demonstrated that it is necessary that the race should not perish now instead of some millions of years hence, as it must. If anyone says that we force a girl to take algebra, but that she is free to accept or reject matrimony as she chooses, then we may honestly deny the assertion, for millions more of girls are forced, in one way or another, to marry than are forced to take any particular subject beyond the most elementary work of the schools. If anyone says that we should improve the conditions that make matrimony so often a failure, then we may reply that many and perhaps all are seeking to do this very thing,—the woman becoming the intellectual companion of the man, the man and woman tending to accept the same social standards, the division of labor becoming more scientifically agreed upon, and the woman assuming her share of the burdens of the state; but in rejoicing in this, we may also say that our failure to progress more rapidly does not mean that we should abolish matrimony, although we cannot say that this will not be done some time, in the centuries ahead of us. Similarly, many of us are doing our best to improve the teaching of algebra, and geometry, and all the other mathematical disciplines, and our slow progress does not mean that we should advocate the closing of the doors of the subject to the youth of our day, although this too may come in the centuries far ahead.

Practical Reform. But what do we teachers of mathematics propose? That question it is impossible to answer with any approach to completeness in these few pages. In general, however, we may say that this is the goal at which we are aiming: That we should cease merely marking time, mathematically speaking, in the arithmetic of Grades 7 and 8, the first two years of the junior

high school; that we should there introduce some definite work in intuitive geometry and in the common uses of algebra, just as the rest of the leading countries do; that in the ninth school year we should show the young people what is meant by logical demonstration in geometry, and what is meant by an algebraic function, by algebraic shorthand, and by generalized arithmetic. It is not necessary that any mathematics should thereafter be required, once the junior high school is well established, with well trained teachers who know the science and can handle this work; but the opportunity should be given for any pupil to continue his mathematics in each year so long as he stays within the academic walls, and the stronger minds will tend to grasp this opportunity as soon as we can adequately prepare the teachers. But until the junior high schools are well established, and until first-class elective courses are offered in mathematics in the senior high school, it would be disastrous to sacrifice the courses in algebra and geometry that we now have, since we have nothing worthy of the student's vigorous mental attack that can take their place. In many places, the present high school will and should endure for years to come, changing gradually for the better in the future as it has in the past. This plan for the junior high school would open the door of mathematics to everyone, just as it should be opened in every other great line of intellectual progress; it would allow everyone to see as clearly as possible the line for which he is apparently fitted; it would give a basis for just the guidance that teachers of mathematics, and letters, and art, and history desire; it would permit of a substantial basis for a series of special superstructures—commercial, industrial, scientific, and so on; it would inflict no real hardship upon any of the youth whom we lead; it would make for better teaching and a far clearer understanding of the great as well as the small things of mathematics, unfettered in the initial stages by any need of what is often a mechanical, soulless cramming for examinations. Given this, and a new light would dawn upon our teaching of mathematics in this country; and with such a curriculum we could bring together the utilitarian features which all agree upon, and the higher life of mathematics, the oversoul, which those who have given the subject the most thought sincerely believe to exist.

Narrow View of Objectives. When, however, we simply hear the wearisome old cry,—the purpose of mathematics is either

(1) its commercial and industrial utility, or (2) that it may produce more mathematics, then we may well inquire if the sole purpose of matrimony is to make money or to produce children. Can we not rise above all this, important as it is, recognizing that the race should be perpetuated and that mathematics should grow, but that this is by no means all of the problem of life or all of the problem of education? Above all, can we not bring the destructionist to see that we cannot leave to the pupil to determine whether or not he is to elect mathematics or language or science or art, until he knows what these subjects mean? In other words, we must require that the pupil take a brief introductory course in mathematics as suggested for the ninth school year, and one in language, one in science, and one in each of the other great branches of human knowledge, since it is only in this way that we put him into a position to judge for himself.

There is little doubt that what has here been said of the oversoul of mathematics will fall like seeds by the wayside, most of it to bear no fruit; and so let it be admitted frankly that we can not, for the state or for the individual, justify the value of this oversoul by any monetary considerations that are measurable by present standards. It must be recognized perfectly that the idea will be considered worthless if judged by certain current methods of investigation. But let it be said with equal frankness that, by the same monetary standards and by the same scales of a limited science, the state should have every mother in this land examined periodically by a body of young people briefly trained in statistical methods, and the moment these same young people find that she has reached the stage when the curve of her cost of maintenance rises above the curve of her productive capacity to the state, then she should be slaughtered and her carcass should be sold for what it will fetch in the public market. It is not a pleasing thought, but it represents a certain type of scientific efficiency.

The Destructionists. It is, of course, well understood that some will continue to demand that all forms of required mathematics shall go, instead of lending a hand to improve its teaching; but this need not disturb us in the least. The pendulum must have its swing, and whether it is forward or back depends simply on our line of sight. It is perfectly evident, to everyone who gives the subject thought, that all that has been so loudly said against geometry may be said with equal energy against letters, against

music, against physics, against history, against the graphic arts, against every other subject in the school curriculum. With even greater force can it often be said against the very courses which these critics give in schools of education all over this country. Mathematics can be justified for the citizen quite as successfully as most of these same courses can be justified for the teacher. If any reader has the slightest doubt about the truth of this assertion, let him write down tonight the exact values, from the standpoint of practical use in teaching, of any course in education that he ever attended—indeed, of any that he ever gave. He will find that these courses have their values, but it may be doubted if the list will be any more impressive than the one which he might also write for mathematics.

The Outlook. Nothing is at its best, not your thoughts nor mine, not your acts nor mine, not your lives nor the life of any one of us. And it is so with mathematics. Everyone knows that mathematics should have a tremendous influence upon the training of the citizen, and yet we so often use our time and energies solely in puttering as to the statistical results of teaching subtraction by this little method or that. While giving all such matters due consideration, can we not take the larger view? Can not the general educator bring his experience to help us in the special fields to make better citizens? Can not we who love mathematics and believe in its larger possibilities bring our skill to help the general educator develop the great elements of life in the souls intrusted to our care? Can not our schools of observation throughout the country recognize these larger opportunities for study in lines that concern those dominant elements that make for a nobler citizenship, as well as in the mere incidents of school life that often claim to be all that research has to offer?

If such a spirit could develop in this country, and if a group of scholarly leaders with a real appreciation of learning and of culture could be found to sympathetically direct this new type of research, then could we, who have joyfully borne the heat and burden of the years, chant our *nunc dimittis,* for then indeed our eyes would have seen the Glory, and then indeed would mathematics, freed from the parasitic growths of ages, take its proper place in the education of our youth.

MATHEMATICS AS AN INTERPRETER OF LIFE *

By W. S. SCHLAUCH
High School of Commerce, New York City

What Knowledge Is of Most Worth? Ever since Herbert Spencer, in his essay on *Education,* first raised the important question "What knowledge is of most worth?" educators have been continuing the discussion which he began. Although they have used varying standards of values, most of them agree that the importance of a subject in the curriculum may be measured by its contribution to:

1. Direct self-preservation.
2. Indirect self-preservation or the earning of a living.
3. Social efficiency and citizenship.
4. The pleasure of the individual.
5. A general understanding of, and an insight into, economic, social and cosmic forces, and problems whose mastery is necessary for the continuing of human progress.

Any subject which can be made to function effectively in the first or second divisions above takes precedence in importance over one mainly useful in contributing to the pleasure of the individual. Thus hygiene and arithmetic are of more fundamental importance to the great majority of American pupils than music, painting, or classical languages. It is my purpose to show by specific illustrations that while mathematics is valuable to a person in the earning of a living it also helps in the attainment of this object by interpreting the economic environment into which the student will be placed on leaving school or college; it is of powerful assistance in building a reliable philosophy; and to those who specialize in the subject, it also leads to the purest and most exalted pleasure of which we human beings are capable. It is, in other words, valuable in all five categories which we have chosen. To prove this I must

* An address delivered before the Mathematics Conference at Teachers College, Summer Session 1927.

first give you my definition of a cultured man, for it is he who is concerned in my discussion. It is this:

A cultured man is one who is at home in the world; to whom the universe and its changes cause none of the terror due to ignorance. In attaining this intimate familiarity with cosmic, economic, and social forces, laws, and changes, mathematics is the one science that makes possible exactness in our interpretations, that enables us to formulate with precision and to predict with certainty.

How Mathematics Should be Taught. While it is true that mathematics as it is often taught in our high schools is an abstract system of symbols that seem to have no relation to anything in the heavens above, or the earth beneath, or the waters under the earth, this need not be the case. It is because teachers are so absorbed in teaching symbols and manipulations that they neglect to point out the meaning of these symbols and their use in deriving practical rules of great utility.

The Interest Formula. To take a specific case, when the student has been taught to solve literal equations in algebra he should be taught to see both the value of the formula and the implications of his results. For example, given the formula for simple interest,

$$i = prt$$

If we let $r = 4\%$ and $t = \frac{1}{360}$ or 1 day,

we have $i = p \times \frac{4}{100} \times \frac{1}{360} = \frac{p}{9000}.$

This gives us the rule:

Rule: *To get one day's interest at 4%, divide the principal by* 9000. Or as expressed by business men: *To get one day's interest at 4%, point off* 3 *to the left and divide by* 9.

Thus, one day's interest at 4% on \$27,900 is \$3.10. To get the interest on \$5,860 for 123 days at 4%, we proceed thus:

90	\$58.60
30	19.533
3	1.953
123	\$80.09

When the student has solved the formula $i = prt$ for p, and has obtained another formula $p = \frac{i}{rt}$, he should be taught to interpret his results in words thus:

Rule: *To find the principal which will yield a given interest in a given time at a given rate, divide the given interest by the interest on one dollar.* It is more important that the student know that $rt = 1 \times r \times t$ is the interest on \$1.00 than that he should be given additional manipulative skill with other symbols, without being able to interpret them or knowing how to use them in the solution of practical problems of life and business.

The Binomial Theorem. When the binomial theorem is developed, most teachers feel that their task is accomplished if the student is able to expand and understand such expressions as

$$(a + b)^n = a^n + na^{n-1}b + n\left(\frac{n-1}{2}\right)a^{n-2}\,b^2 + \ldots.$$

while, as a matter of fact, the student usually has very little interest in the process itself, except that derived from his desire to be able to pass the examination at the end of the term. Yet all students in the high school know about bonds, interest, and investments, and hear such topics discussed at home. Their interest in the binomial theorem would be tremendously increased if the teacher were to introduce it as a means of easy solution for compound interest problems. Let the teacher by using arithmetic examples derive the formula for compound amount,

$$A = P(1 + i)^n$$

and then show the students that any compound amount may be easily calculated, provided we know the value of the accumulation factor

$$(1 + i)^n.$$

Let the teacher then show the difficulty of this calculation by arithmetic and the facility of it by the binomial method, and the students are eager to master the new device. Thus, the compound amount of \$500 for 5 periods at 2%, the savings bank rate, is

$$A = 500\ (1.02)^5$$

Now $\quad (1 + i)^5 = 1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5$

Substituting, $\quad 1.02^5 = 1.1040808032$

Therefore, $\quad A = \$500 \times 1.1040808032 = \$552.04$

The entire process is performed mentally and, when a certain amount of skill is developed, with great ease. The uses of the binomial formula such as in finding roots may also be pointed out to advantage.

The Theory of Exponents. When the theory of exponents is taught, most students in our high schools wonder why they should be required to know that

$$a^{-2} = \frac{1}{a^2}, \text{ and that } x^0 = 1.$$

It seems to me that if this knowledge were applied to a useful purpose, namely, the mastery of the theory of logarithms and computation by logarithms; and if then the use of the slide rule as an application of logarithms were given, our students would find the intermediate algebra course a fascinating one, with sidelights, generalization, and interpretations of what was obscure in arithmetic, instead of the abstract system of symbols it now is in the hands of most teachers. Logarithms can be applied in calculations dealing with business, accounting, and the economic world. Amortization, sinking funds, depreciation, and bond valuation are some of the topics that can easily be mastered by one who has a knowledge of logarithms and of geometric progression. Yet we emphasize the abstract theory of quadratic equations, the remainder theorem, and the graphs of functions, without pointing out the practical implications and interpretative value of the most important parts of algebra and geometry.

The Use of Graphs. The same subject, which is presented to one student as an abstract system of symbols, may be so presented to another that he regards it as a powerful instrument for solving problems in which he is interested, and as an explanation of the phenomena by which he is surrounded. Graphs, as presented in most conventional textbooks, and as taught by the majority of teachers in our high schools, deal with the mere plotting of functions of x and y. Yet, if the subject is approached through statistical graphs and graphs for calculation, the student may easily be taught that certain problems connected with building, railroad timetables, points of maximum net return, and the like, may be solved by graphic methods more easily than in any other way.

As a simple illustration, consider the following problem: A total of 151 M. bricks are needed for a building, and they will be used as follows: 4 M. per day for the first 4 days, 10 M. per day for the next 6 days, and 15 M. per day for the last 5 days. As a matter of economy, it is desired to have the deliveries made at a uniform rate per day. Find the maximum storage capacity needed to allow uniform deliveries over the 15-day period.

It is a very simple matter to plot the graphs of total number used, and total deliveries, as shown in Fig. 1, and by measuring ordinates between these graphs, find the maximum number of bricks stored.

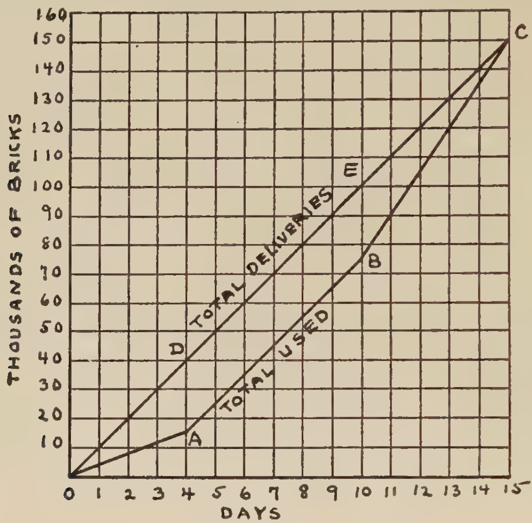

FIG. 1

By measuring the ordinate differences $A\,D$ and $B\,E$, we find that $B\,E$ is the greater and that the maximum storage capacity needed is approximately 10,400 bricks. A graph makes this clearly evident. The varying ordinate differences between the line "Total Used" and the line of "Total Deliveries" show the growth and disappearance day by day of the pile of stored bricks.

The Use of Geometry in Calculation Charts. The interest of students in plane geometry is tremendously increased when they find that it gives them an insight into certain calculation charts used in business, enables them to understand instruments of precision, and clarifies for them geography and astronomy. Any normal child is blessed with natural curiosity—that heritage of the evolutionary struggle during which not to comprehend the environment and its dangers meant death. Children take joy in mastering knowledge which they can see has some relation to the phenomena of their lives. It is only the mass of abstract material in a dull curriculum, unpedagogically presented, that finally kills the desire to learn. For example, when students have learned to prove the congruence of triangles, it gives them a thrill of interest to find that

this knowledge enables them to understand the principles involved in the construction of simple alignment charts, such as paymasters' wage alignment charts. This fact can be seen from the following:

In Fig. 2, we have axes AA, BB, and CC at right angles to AB, such that $AC = CB$, and the scales of numbers on AA and BB are so placed that the same length on each has a unit value. If on the

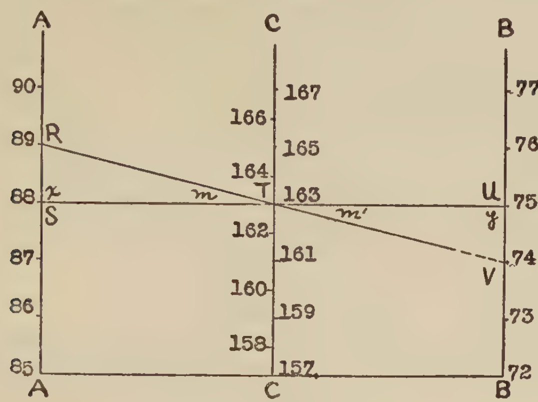

Fig. 2

level of any two numbers on the outer axes, whose join is paralleled to the base AB, we place on the C-axis the sum of these numbers, we have an addition-subtraction chart within the range of the numbers employed. To prove this we use the theorems of the congruence of triangles. Thus in Fig. 2, since the join of 88–75 was numbered 163 on CC, if we join any two numbers whose sum is 163 the line will pass through this same point.

Proof. Draw a line from 89 on AA to 163 on CC and continue the line to the B-axis. In triangles RST and TUV thus formed:

$\angle x = \angle y$	(Because they are right angles.)
$ST = TU$	(If a series of parallels intercept equal segments on one transversal, they intercept equal segments on any other transversal.)
$\angle m = \angle m'$	(Because vertical angles are equal.)
$\therefore RST \cong TUV$	(Because two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.)
$\therefore UV = RS$	(Because corresponding sides of congruent triangles are equal.)

Therefore the prolongation will strike the point 74 on *BB*. Since only *one* straight line can be drawn between two points, *any line joining V and R must coincide with RTV, and therefore pass through T.*

Wage Alignment Charts. This same principle can be made so clear to the student that he will be able to construct a wage alignment chart for calculating wages due when time rate and overtime rate are given, as illustrated by the fragment shown in Fig. 3.

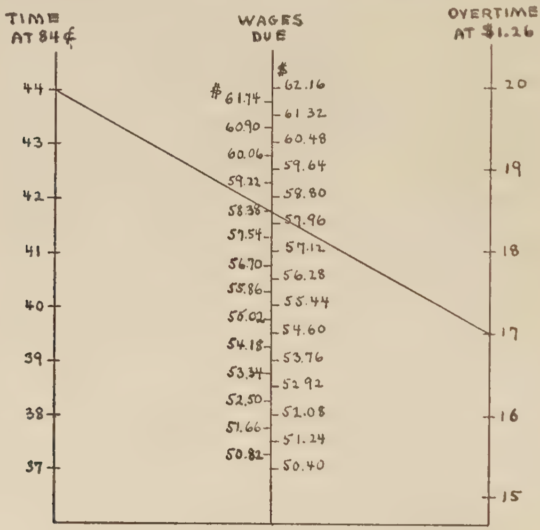

Fig. 3

In Fig. 3 it is seen that if we join 44 hours of time with 17 hours of overtime, the wages due are $58.38, the rates being $0.84 and $1.26 for time and overtime, respectively. In this chart, 3 hours of time occupy the same length on the time axis as 2 hours of overtime on the overtime axis. Since the rates are 2 : 3, the *values* represented by equal distances on the outer axes are equal; hence the scale of values on the center axis shows double values, as in the simple addition-subtraction chart.

The Use of Geometry in Geography and Astronomy. Geometry can be made to throw light on geography and astronomy, giving the student a more intimate knowledge of the world in which he lives. Consider the simple application of geometric theorems to Fig. 4.

This figure represents the position of the earth at noon on December 21 with relation to the sun's rays SS'. NS is the earth's axis, EE is the equator. Since the sun's rays to the earth are practically parallel, an observer at O sees the sun along $OS'S$, and its angle of

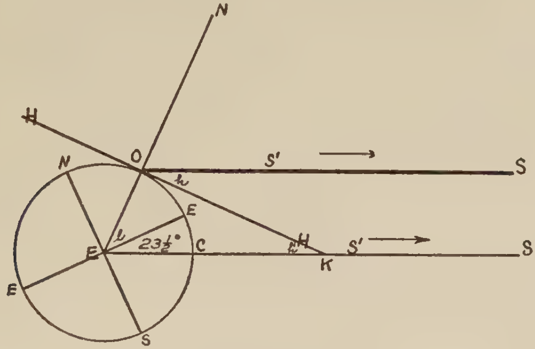

FIG. 4

elevation above his horizon is h or SOK. An observer at C, 23½° south of the equator, sees the sun at noon along $CS'S$ or in his zenith, at the same instant that the observer at O sees it $h°$ above his horizon.

Now, any student of plane geometry, in the second semester, can prove:

(1) That $\angle SOK = \angle OKE$, or that $\angle h = \angle h'$

(2) That if l is the *latitude* of observer O, $23\frac{1}{2}° + l + h = 90°$

(3) $\therefore$ That $l = 66\frac{1}{2}° - h$

Students thus understand that an observation taken at noon, which will determine the angle of elevation h of the sun above the horizon, will enable one to find his latitude.

The Geometry of the Sextant. The geometry of the sextant by which this angle is measured should then be presented, and in the mathematics club a very interesting session can be devoted to an explanation of it. A homemade sextant has been made by the writer at a cost of less than a dollar for materials, and its readings are surprisingly accurate. Angles can be read to the nearest quarter degree. Rotating mirrors for measuring horizontal angles, the plane table, the pantograph, and the like, give the student an insight into the methods by which men measure space relations, and help him to interpret the world in which he lives.

Problems of Finance and Investment. If time permitted, I should like to consider a series of problems and calculation which involve nothing more difficult than logarithms and geometric progression, but which give the student a real insight into the problems of finance and investment that mean so much in our economic environment. They would constitute the subject matter in a course of more vital importance to the student than the conventional mathematics of the last years in the high school, and would enable a student to understand the world in which he must make his living much better than nine-tenths of the subjects now offered in the curriculum. I refer to such subjects as *installment payments with interest, amortization and borrowing, sinking fund calculations, and bond valuation.* It is also possible to teach a high school student the elements of statistics, to study trends and cycles in business, to apply graphic methods, and in general to open up a subject so as to make it fascinating and profitable.

Such a course is actually given in the High School of Commerce of New York City. It is a tremendous help to the students in their work in economics and accounting and their subsequent studies in finance and investments.

Value of Mathematics in a Higher Sense. But there is a still higher sense in which mathematics becomes an interpreter of life and the world. If it is properly taught, with inspiring sidelights and comments by a live teacher, there should emerge in the mind of the student the conception of an ordered, lawful universe in which all phenomena yield to quantitative investigation, a universe in which the reign of law is absolute, and in which all phenomena in this space-time world fall into order in a cause and effect nexus. As Bertrand Russell says:

> Mathematics, rightly viewed, possesses not only truth but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection, such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not only to be learnt as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the ideal and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the

creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs.

It is, moreover, true that mathematics in our day has solved some of the age-old metaphysical problems, and revealed the nature of reality so conclusively that the attempts of the old metaphysicians seem by contrast puerile. While our philosophers are still discussing the paradoxes of infinity, solved by Weierstrass, Dedekind, and Cantor, the mathematicians press forward into the realms of hyperspace and relativity. It is not too much to say that the mathematicians are at least a half century ahead of the philosophers in their interpretations of the problems of philosophy. The problems of the infinitesimal, of the infinite, and of continuity were solved definitely by the three great mathematicians just mentioned. Einstein and others are pressing forward to new triumphs.

In many classes in philosophy, for example, the old puzzle of Achilles and the tortoise is still being discussed. Zeno reasoned that Achilles can never catch the tortoise because: "At any definite instant of time the tortoise is in a definite position, and Achilles is in a definite position. Therefore the number of positions occupied by the tortoise is equal to the number of positions occupied by Achilles, since the time consumed by each in the race is the same. But, if Achilles were to overtake the tortoise the positions occupied by the tortoise would be only a *part* of those occupied by Achilles. Since a part can not equal the whole, Achilles can not in reality overtake the tortoise." Some philosophers in discussing "the ultimate nature of space" are still puzzling their classes with this old paradox of Zeno, not knowing that the mathematicians have solved the puzzle. The explanation is that since space and time are both infinitely divisible, the *number of positions* occupied by Achilles and by the tortoise are both infinite, and the axiom that "the whole is greater than any of its parts" is true of finite positive wholes but not true in the realm of infinity. In infinity a *part may equal the whole,* as Bertrand Russell and Professor Keiser have shown in recent non-technical essays. So it follows that although the positions occupied by the tortoise are only a part of those occupied by Achilles, their numbers are *equal,* as the equal times occupied indicate, and Achilles *does* overtake the tortoise.

Mathematics transforms, sobers, and exalts, as well as glorifies our view of reality. Old superstitions and childish beliefs as to the structure of things are taken from us, but in their place is given a

profound, satisfying, lawful vision. The complaint of the tender-hearted over the loss of the naïve vision is thus voiced by Poe:

> Science! true daughter of Old Time thou art,
> Who alterest all things with thy peering eyes,
> Why preyest thou thus upon the poet's heart,
> Vulture, whose wings are dull realities?
> How should he love thee, or how deem the wise,
> Who wouldst not leave him in his wandering
> To seek for treasure in the jeweled skies,
> Albeit he soared with an undaunted wing?
> Hast thou not dragged Diana from her car,
> And driven the Hamadryad from the wood
> To seek a shelter in some happier star?
> Hast thou not torn the Naiad from her flood,
> The Elfin from the green grass, and from me
> The summer dream beneath the tamarind-tree?

My answer to Edgar Allan Poe is this:

> Science, most favored of Old Time thou art,
> Who seest all things with wide-opened eyes.
> Thou thrillest mightily the poet's heart,
> Eagle whose pinions touch the lofty skies.
> How he should love thee, who hast made him wise,
> Who thrilled his heart, and taught his voice to sing
> Of majesty and wonder in the skies
> Wherein thou soarest on undaunted wing.
> Hast thou not gathered wonders from afar,
> And banished superstition from the wood,
> And lifted craven fear from every star?
> Hast thou not conquered death in fire and flood,
> Opened the book of nature wide to me,
> And shown the Reign of Law in earth and sea?

THE REALITY OF MATHEMATICAL PROCESSES

By E. R. HEDRICK
University of California at Los Angeles

Purpose of the Discussion. The attempt to appraise values in all our curricula is timely enough. Moreover, it is vital because it will correct the faults of our system, both present and future. The purpose of this chapter is to direct attention to phases of mathematical values that are overlooked in many surveys of this field.

The greater part of modern study of values in mathematics has been directed toward the discovery of precise criteria for the amount of application of the various mathematical facts and skills in the work of the world and in the life that the student is to lead after his schooling. Such practical applications are indeed of utmost importance. I shall be the last to decry them. Let us by all means know which facts, which skills, are of real use. Let us drill on those that are applied, and not on others. Let us determine such values as precisely as it is at all possible to determine them. Let us not stint praise to those who do determine such values.

While I do not condemn, but rather laud, the search for practical values of given facts and skills, I desire to turn attention for a little while to a closely associated but more neglected field. Are there not other values just as real as those that come from specific knowledge of isolated facts or from specific skills? Too often, it seems to me, such values are denied or are overlooked because the application of facts and skills is the more obvious.

Mathematical Processes. For lack of a better word, I have chosen the word *processes* to cover the field I have in mind. I admit freely that this word has to cover a rather wide range. It is somewhat indefinite. It must not be vague, however; nor must this degenerate into nebulous discussion of a general training of the mind. For, aside from facts and skills, there are in mathematics numerous ways of thinking, ideas, concepts, methods of procedure,

that cannot be classed either under the head of factual knowledge or under the head of skill in a technique.

Numerous examples will suggest themselves to everyone familiar with mathematics. I shall mention several presently. While it is difficult to give a sweeping definition of all such processes, and equally difficult to mention all examples, the type will be clear from a few such instances.

Real Values in Mathematical Processes. To run ahead of an extremely logical procedure for the moment, assured that all know many examples, may I ask now whether real value—value for the most practical side of life—does not arise from what I am calling the *processes* of mathematics?

Are the values less real, in the processes which occur to your minds, than the values that arise from knowledge of facts, or from technical skills? In each instance that I quote I shall ask you to weigh the value in actual life. I shall ask also if the process concerned is one that can be or is commonly acquired by people without training of any sort, and whether a study of mathematics in particular is a usual or a necessary means for its acquisition. It remains to be seen whether this is true.

It will be important to divorce the process itself from the facts usually associated with the process, to generalize from particular cases, as is common otherwise in mathematics, in order to avoid a confusion of ideas. It is very easy to think only of individual cases, of special facts, of given skills. This is, indeed, the danger. If we allow our attention to center on a special case, on a given fact, on a particular skill, we are again on the field which has been discussed most often. We shall be in danger of abandoning the process itself, as has been done by many before us.

Possibility of Transfer. Before I proceed to examples, let me mention the possibility of *transfer* of training in the field of which I am speaking. Are processes themselves subject to transfer? May a training in the use of a process enable one to apply that process in a different set of activities? It seems to me that this may be the true key to a great part of the theory of transfer. Can a given fact be transferred? Can a particular skill be applied in any other than its original setting? These transfers seem far more problematical than is the transfer of a process. Perhaps I might even define a process as that part of training which is subject to transfer to a totally different set of ideas. If this be valid, it is through

training in processes that we may hope for transfer. It may be that here, rather than in facts and skills, the truest values lie.

I. Specimen Processes

Precise Statements and Precise Reading. The simplest and most sweeping of processes are *precise statements* and precise reading of statements. Any teacher will know that precision of statement requires training, experience, and repeated illustration by particular instance after particular instance. The facility in reading a statement precisely, so that all of its meaning is apprehended, is one of the most vital things we teach. May I rehearse the questions I have asked above? The answers are too obvious for statement. Is this process one that requires some training? Is it useful in life? In every life? Does the average individual acquire it commonly without training? There are, indeed, other subjects in which precision of statement and precision of reading are desirable. I believe, however, that there are few other subjects in which minute differences of wording give rise to such errors and to such total misunderstandings, and there are few subjects in which care and direct training in precise statement and precise reading are commonly part of the regular procedure.

Generalization. Another quite general process is the very one that I have used in my own argument. It is the process of *generalization* from particular instances to an abstract or general idea. Thus, as I have remarked, the idea of what I have called a process is itself a generalization from numerous individual cases. Need I ask again the same crucial questions? I may at least point out again that it is in mathematics more than in any other subject that *generalizations, abstractions,* are commonly presented. As in the case of precision, so also here in the case of generalization, the possibility, nay the moral certainty, of useful transfer to other fields is unmistakably clear.

Necessary and Sufficient Conditions. As a third specimen of wide application, I may cite the ideas of necessary and sufficient conditions and the difference between conditions that are necessary and those that are sufficient. The resulting confusion among those not properly trained is notorious, and this confusion is certainly transferred to every field of thought, from cookery to politics. It is in mathematics that these thought-processes are definitely trained, and it is in mathematics that instances are given which

bring the young student to a realization of their vital character. That the thought-process concerned in the distinction between necessary and sufficient conditions does transfer to other fields is hardly debatable. Consider, for example, the oft-debated question of a study of mathematics by girls. The transfer of the facts and skills of mathematics to the most common tasks of women is, indeed, debatable. But consider the transfer of this thought-process. Have you not known women who thought flannel *necessary* because it is sufficient? Who thought lard necessary until shown that other fats were sufficient? In many of the homeliest activities, such distinctions, and this very thought-process, are a vital guide to swift and economic action. My claim would be that a student trained in mathematics to distinguish necessary from sufficient conditions would normally inquire in any situation whether a thing known to be sufficient were or were not a necessity.

II. Processes of Special Character

Functional Thinking. In what precedes I have mentioned processes of the widest possible scope. Many of more special character will occur to you. In several, the mathematical character is so pronounced that no one would expect the desirable training to be acquired in studying any other subject. One such subject which has received extensive attention of late years is the functional relationships between quantities. That quantities are often interdependent, and that the effect of changes in one quantity upon the values of another should be carefully considered, is a germane part of any sound mathematical teaching. Instance after instance is discussed in every course on mathematics, and emphasis should be laid upon each of them. Not all relations between quantities are straightforward proportions. In few other courses are these ideas presented and insisted upon. Their transfer to life activities is again incontestable. Lack of training is evident in persons in every walk of life, since every person deals with quantities; and many are they who fail because they do not grasp the vital relations between quantities that enter in their lives. Thus compound interest mounts higher than they imagine, if they have not thought of its relations to time and rate. Costs of transportation depend on distance, but also on the number of times materials are handled. The habit of thinking about relations between quantities is a thought-process that transfers to all activities of life

more readily than does any given fact or any known technical skill.

Effect of an Error in Measurement upon a Computed Result. A more specific instance that is really a special case of functional thinking is the thought-process which occurs when we estimate the effect of an error in a measurement upon a computed result. Thus an error of one inch in measuring the side of a ten-foot square causes a resulting error obviously nearly two square feet in the area which is computed by the formula for $(a + b)^2$. Such errors are common in everyday life; their effects are frequently much larger than the unwary suppose. A little training in thought about the effects of errors will quite surely transfer to a great range of life experiences.

Logical Proof. Certainly I should not fail to mention logical proof as one of the great processes vital to mathematics and taught more consistently in mathematics than in any other subject. Deduction of facts from other presupposed facts, the axiomatic foundation, the insistence upon reasoning as against instinct,—such habits of thought once learned persist. A delightful little book, *Thinking about Thinking* by Professor Keyser, grew out of his own mathematical training, but it deals with the wider field of all thinking, and it has found a surprisingly large circulation and a profound degree of appreciation in wholly non-mathematical circles. This is the best evidence of the possibility of transfer of such training.

But I might mention hundreds of examples of such processes: Reasoning by exclusion. Reasoning by reduction to absurdity. Mathematical induction. Maximum and minimum. Averages. Graphic representation. Rate of change. Tables of numerical values. Precise definition of terms. These and many more are strictly mathematical processes of constant occurrence and of wide application in life activities.

Idealization. One final instance that I would emphasize is that of *idealization.* We deal with idealized concepts frequently. A material straight line does not exist; it is an idealization from realities. Points themselves are ideal. Circles, surfaces, all the thousand and one concepts of geometry require the mind to conjure up a pure ideal from the crassness of reality. This is a process that must be learned. We idealize many concepts in the world of life. Idealizations run all the way from that of the curve called *parabola* to that of the color called *pink,* from that of *horsepower*

to that of *heaven.* Some are distinctly mathematical, as is horsepower. Others, less so, are the easier to grasp and to hold for one who often idealizes—to whom the process of idealization has become habitual.

Summary. May I then state conclusions which seem valid to me? There are, it seems to me, a great many processes in mathematics which, quite aside from any facts with which they are commonly associated, quite aside from any definite technical skill, are very real and very important. Some of these are shared by mathematics with other subjects, but those that I have mentioned are, and of a right ought to be, emphasized and illustrated in mathematics more than in any other subject. These are not hazy generalities, however. I have not talked in vague terms of a general training of the mind as a whole.

All those that I have mentioned, and many more, occur prominently in the life of the world about us, in the life activities of all our students. Such processes, far more than single facts and given skills, find true application in far wider and more vital fields of human living and of human thinking.

It is for this reason that transfer of training seems to me to be more possible with respect to processes than with respect to facts and skills. In certain instances, as I have pointed out, transfer is actually inevitable and wholly incontestable.

That this is true seems to have been overlooked by many who sought values alone in facts learned and in skills acquired. A recognition of the truth of my contentions—if truth they be—will necessitate almost complete revaluation of topics in arithmetic, algebra, geometry, since values have been assigned largely on the basis of facts and skills, and on the possibility of transfer of facts and skills. Is it not possible that the true explanation of the fact that the transfer of mathematical training is now known to be more certain than was supposed only a few years ago, may be that transfer occurs in what I have called *processes?*

A tale of my student days in Paris recurs to me. Then I knew well many French students. They had just passed through what was to all Frenchmen a most harrowing period of public stress caused by the famous Dreyfus trial. I heard it discussed many times. One phase which interested me and fellow students of mathematics centered on letters admittedly written by Dreyfus.

These were said by his enemies to contain cipher messages, the assertion being based on the fact that the letters of the alphabet did not occur in them in the proportions in which they are known to occur in average French prose. Among the defenders of Dreyfus was Henri Poincaré, the greatest mathematician of France. He tried in his humble professorial style to convince stubborn and hostile lawyers that the most probable proportion of the letters would be highly improbable. Can you imagine with what sarcasms the hostile opponents flayed this professor in his unworldliness? But for that flaying, justice returned and triumphed. The scholarly world of Paris revolted at the slight placed upon scholarship, and at the lack of reverence shown to Poincaré. Another, more accustomed to debate, familiar with the world of lawyers and courts, then only a professor indeed, Painlévé, since Premier of France, made even hostile lawyers see that the most probable proportion was highly improbable in a given case. "Give me the works of Racine," he thundered, "and I will show you that he, too, by your foolish tests is a traitor, for the works of Racine, like the letters of Dreyfus, do not show the most probable distribution."

Such is our case. I spoke at first, in our accustomed way, of the "practical" applications of mathematics,—of the facts learned, of the skills acquired. These we call by habit the practical things. Thought-processes, ways of thinking, procedures—these we put to one side when we are speaking of the "practical" applications. May it be that the most practical things are those which are not practical? Is this a contraction of terms? Let me recall that the most probable distribution of letters is very improbable.

Have we missed the most practical of all things in our strenuous search for the practical? What can apply more directly to life? What can transfer more frequently or to a wider field of activities? Is this, then, not practical?

Shall we abandon processes and ways of thinking in favor of fact-learning and skill-acquisition because these latter seem on the surface the more practical? What if the fact, the skill, does not apply to life? What if it does not transfer? May not the process, the way of thinking, transfer to life when the fact, the skill, does not? Which, then, is the more practical—the "practical" one? May not the impractical prove the more practical?

What shall it profit a man if he learn every fact, and acquire every skill of mathematics,—if he loses the soul of the subject?

DEVELOPING FUNCTIONAL THINKING IN SECONDARY SCHOOL MATHEMATICS

By E. R. BRESLICH
School of Education, University of Chicago

Emphasis on the Function Concept. The importance of using the study of mathematics as a means of training in functional thinking is being recognized more and more. It has long been emphasized by leading mathematicians and has been widely discussed in mathematical journals, teachers' conferences, and committee reports. The suggestion has been offered that in the reorganization of secondary mathematics the function concept be made the unifying principle. Professor Klein, of Germany, refers to it as the "soul" of mathematical instruction. Indeed, without functional thinking there can be no real understanding and appreciation of mathematics.

An Analysis of Current Texts. Curiously enough, although the demand for emphasis on functional thinking has been advocated for a long time, authors and teachers as yet pay little attention to it. Objective evidence for this statement is not difficult to find. An analysis of several widely used algebras and geometries revealed that although on almost every page opportunities for training in functional thinking are offered no systematic use of them is made. Four texts were examined in this study. Two of them, which will be designated by A and B, are of the traditional algebra type. According to the authors' preface, a special attempt is made in text B to bring out functional relationships. Texts C and D are of the conventional plane geometry type. Since the books used in this study give typical presentations of algebra and geometry, the findings indicate what is general practice.

The algebraic problem material that lends itself especially to training in functional thinking was classified according to the following types:

1. Verbal problems that are solved by means of equations and algebraic relationships not of types 2 to 4 listed below.

2. Problems of physics dealing with the laws of uniform motion, falling bodies, levers, and the like.

3. Problems involving business applications and leading to formulas of percentage, interest, and the like.

4. Problems expressing number relations and calling for the value of one of them. This includes the so-called puzzle problems found in algebra texts.

5. The total of types 1 to 4.

6. Problems in spatial relations employing formulas of geometry such as areas, volumes, and perimeters.

7. Graphs.

Similarly, the geometric material was classified under several headings:

1. Congruence of figures.
2. Inequalities.
3. Changes in figures.
4. Changes in formulas.
5. Proportional magnitudes.
6. Mensuration.
7. Trigonometric functions.

The results of the study are summarized in the accompanying table. The percentages express the ratio of the number of cases in which attention is called to functional relationships to the number of opportunities actually offered. Thus, the ratio $\frac{21}{1048}$ means that out of 1,048 opportunities the author made use of 21.

Text A practically disregards opportunities for training in functional thinking except where they are offered in connection with the teaching of graphs. The author seems to associate the function concept almost entirely with graphic representation. Text B likewise gives emphasis to the function in the teaching of graphs, but also calls attention to functional relationships in the study of formulas. Thus, even when authors express the intention of stressing functional thinking in algebra, they fail to make use of at least 1 out of 9 of the opportunities offered. This is due partly to failure to plan the work carefully.

In geometry the situation is even less satisfactory than in algebra. Although by far the greatest part of plane geometry deals with relationships, one of the two textbooks examined disregards the function concept entirely while in the other it is associated with the changes taking place in the formula corresponding to changes in geometric figures.

TRAINING IN FUNCTIONAL THINKING IN ALGEBRA AND GEOMETRY

Textbooks in Algebra	Problems leading to algebraic equations and formulas	Problems using formulas of physics	Problems using formulas of business	Problems expressing number relations	Total percentage for problems and formulas	Problems in spatial relations	Graphs
A	$\frac{21}{1048} = 2\%$	$\frac{0}{17} = 0\ \%$	$\frac{6}{761} = .8\%$	$\frac{0}{804} = 0\ \%$	1 %	$\frac{0}{216} = 0\ \%$	$\frac{426}{546} = 78\%$
B	$\frac{25}{1225} = 2\%$	$\frac{52}{171} = 30.4\%$	$\frac{211}{397} = 23.5\%$	$\frac{24}{434} = 5.5\ \%$	11.4%	$\frac{102}{483} = 21.2\%$	$\frac{722}{881} = 82\%$

Textbooks in Geometry	Congruent figures	Changes in figures		Proportionality	Mensuration	Inequalities	Trigonometric ratios
		Surface changes	Changes in formula				
C	$\frac{0}{837} = 0\%$	$\frac{0}{630} = 0\ \%$	$\frac{0}{60} = 0\ \%$	$\frac{12}{1542} = .77\%$	$\frac{60}{546} = 11\%$	$\frac{0}{396} = 0\ \%$	$\frac{0}{97} = 0\%$
D	$\frac{0}{729} = 0\%$	$\frac{140}{778} = 18\ \%$	$\frac{38}{175} = 21\ \%$	$\frac{37}{2858} = 1.3\ \%$	$\frac{0}{1200} = 0\%$	$\frac{0}{408} = 0\ \%$	

The table should read as follows: In Text book A 21 out of 1048 opportunities offered (i.e., 2%) were actually made use of to give training in functional thinking, and so on.

Opportunity for Studying Functional Relationships. In the following pages it will be shown that functional relationships may be studied in all the mathematical subjects taught in the secondary school from arithmetic through trigonometry. As long as we restrict the teaching of functional relations to certain topics, such as graphs, changes in geometric figures, ratio, proportion, and variation, we cannot expect satisfactory results. No opportunity should be missed to call the pupil's attention to relationships wherever they occur, if functional thinking is to be truly cultivated.

A critical examination of the instructional materials of secondary school algebra reveals the topics that are expected to deal with quantitative relationships, and in which attention could and should be called to such relationships. A brief discussion of these topics follows.

Relationships Stated in Words. In the verbal problems of algebra the pupil is frequently expected to formulate the mathematical law which expresses the relationship involved. This is common in applied problems of which the following illustrations are typical examples: If a train travels at a uniform rate, the distance traveled depends upon the time; the perimeter of an equilateral triangle depends upon the length of the side; the interest on a sum of money depends upon the time; the weight of an iron rod depends upon its length; the cost of sugar depends upon its weight.

Statements like those above offer opportunity for functional thinking by discussions of the factors involved in the problems and of the nature of the relations between them. Teachers, however, are likely to ignore the relation aspect in verbal problems and to pass directly from the statement to the mathematical law and then to the solution of the equation. The pupil is thereby deprived of training in functional thinking.

Relationships in Tabular Representations of Numerical Facts. Numerical facts are commonly presented in the form of tables. It is then necessary for the reader to interpret the facts presented in the tables to appreciate their meaning. Some tables contain facts that are related to each other by precise mathematical laws. In others relations exist which cannot be expressed precisely. In still others there exists correspondence, but no relationship. In all cases there are opportunities for studying the facts from the standpoint of functional relationship, that is, of dependence and

correspondence. When a relation is stated in words the pupil may have to construct his own table to make the relationship clear to himself. Various types of tables are described below.

1. Tables in which the corresponding facts are not related by precise mathematical laws. An illustration of this type is the table showing various items for which taxes are spent. The facts are usually arranged as follows:

For Library	$ 1.25
For Parks	10.00
For Pensions	2.00

At first glance the facts in the table seem to be entirely unrelated. However, the table is an example of the general principle on which all tables are constructed, that is, the arrangement of two series of facts placed so that the facts in the same row always correspond to each other. Furthermore, to appreciate the meaning of the table the pupil may rearrange the numbers in increasing order, or he may make comparisons between any two items by finding their ratio. He may reclassify the items under such headings as expenditures for education, amusement, improvements, and the like, and then by means of ratios work out relations between these classes.

Other familiar examples of this type of table are those giving the time of day and corresponding temperature; certain dates and the corresponding population of a country; the price of wheat for various months; and the amount of light used by the average family for each month of the year. All of them clearly exhibit correspondence, but the teacher may easily use them to teach functional relations by calling attention to changes, fluctuations, and dependence.

2. Tables in which precise relations exist but are not stated. To this class belong the tables used for computational purposes; for example, the tables of roots, powers, logarithms, and trigonometric ratios. When they are used simply as labor-saving devices without attention to the relations that are involved, the pupil may have but a limited conception of what such tables mean. He may know that the facts in the same row of a trigonometric table correspond to each other, but he may go no further. On the other hand, if his attention is called to the way the numbers in the table change, he will observe that the sine of an angle changes continuously from 0 to 1 as the angle changes from 0° to 90°. An understanding of

these changes gives him a much better conception of the table and aids him in using it. Thus, having observed that the sine of an angle changes more rapidly between 0° and 45° than between 45° and 90° he is able to locate in the table any given value of the sine function more quickly than one who has given no thought to the matter.

3. Tables containing facts determined from scientific studies. Well known examples of this type are the tables giving cost of life insurance and age of applicant, heights of children and corresponding ages, weights of boys and girls for given years, and so on. A study of the changes of the numbers contained in the tables raises and answers many interesting questions, and offers opportunities for training in functional thinking.

4. Tables constructed by the pupil from precise mathematical laws. Here should be mentioned time-tables, tables of longitude and corresponding local time, tables containing distances traveled in given periods of time, and interest tables.

5. Tables made by the pupil for the purpose of discovering laws. Illustrations of such tables are the tabulation of corresponding diameters and circumferences which leads to the discovery of the circumference formula, and the tabulation of the successive terms of a progression for the purpose of finding a formula that will give any required term of the progression.

6. Tables giving a series of values of polynomials corresponding to given values of the independent variable. If the pupil arranges in tabular form a series of values of the variable and the corresponding values of the polynomial, a study of the table will give him a clear picture of the behavior of the polynomial. In most of the so-called evaluation exercises of algebra the functional aspect is completely lost sight of because the exercises call only for values of the function corresponding to isolated values of the variable.

7. Tables used in graphic representation. In representing numerical facts graphically the making of a table of corresponding facts is usually the first step. A study of the data of the table reveals the range of the variables, the size of the unit to be used in making the graph, and other important facts, and results in a clear picture of the behavior of the function even before the graph is made.

The discussion above has shown that tabular representation is widely used in the study of mathematics and that, without emphasis

on the relationships involved in the tables, the pupil may be deprived of much valuable training in functional thinking. Indeed, he may fail to appreciate the full meaning of tabular facts.

Using the Formula to Represent Numerical Facts. In solving problems the formula is frequently taught as an "abbreviated rule" of a verbal statement. As such it simply states a procedure to be followed. Thus the formula $A = bh$ directs the pupil to multiply the base by the altitude when he wishes to find the area of a given rectangle. He is likely to follow the direction without any knowledge of the fact that the formula expresses dependence or relationship between the variables A, b, and h. Indeed, some textbooks give three formulas, one for finding A, another for finding b, and a third for finding h. Accordingly, the pupil must learn three formulas as three distinct "cases," but he gives no thought to the way they are related to each other unless the teacher calls attention to the relationship. The "case method" tends to obscure relationships. Tests given by the writer show that best results are obtained if the pupil in solving problems about area, interest, circumference, percentage, and so on, thinks of *one* relation only in each case, as seen in $A = bh$, $c = 2\pi r$, and $i = \frac{5}{100} pt$. If he understands that relation thoroughly, then, in the solution of a problem, he will always go back to it and solve it for the required number rather than choose the "case" which fits a particular problem.

Evaluation Exercises with Formulas. Evaluation exercises with formulas offer further opportunity for functional thinking. However, this opportunity may be entirely lost when the mind of the pupil is centered wholly on this sort of work, because the formula is thus looked upon as a basis for computation only. A study of the changes produced in the formula by changes of the values of the variable should go hand in hand with the calculation of the values. Dependence and correspondence should always be made to stand out clearly.

Formulas dealing with laws of physics, such as uniform motion, falling bodies, and the lever, are naturally taught with some emphasis on relationship. But even here, the solutions of applied problems may amount to little more than substitutions in the formulas followed by solutions of the resulting equations, and the pupil may be entirely ignorant of the relation expressed by the formula,

To help him acquire a real grasp of the algebraic formula both aspects must be brought out in teaching, the relationship being fully as important as, or even more important than, the numerical calculation.

Functional Thinking through the Study of the Equation. As in the formula, several aspects of the equation are to be emphasized if real understanding is to be attained. Some equations are statements of mathematical laws, others equate two expressions denoting the same number, and still others are symbolic translations of verbal statements. The pupil may think of the equation simply as a question asking for the value of an unknown number. For example, to find the angles of a triangle if the first is one-half as large as the second and the third four times as large as the first, he may solve the resulting equation, $\frac{1}{2}x + x + 2x = 180$, without giving any thought to the relation involved. On the other hand, he may think of the statement $a + b + c = 180$ as a relation between the three angles of any triangle. As soon as the sizes of two are known, that of the third is fixed. Such considerations call attention to the functional aspect of the equation. In the same manner all equations may be considered as expressing relations. For example, in the equations $5x - 16 = 0$, $2x^2 - 5x + 1 = 0$, the left members are more than polynomials whose fixed values are zero. They are variables having an infinite number of values, although in this case interest happens to be centered upon the particular value, zero.

The Study of Algebraic Polynomials. As has been pointed out in connection with the equation, complete understanding and appreciation of the polynomial is possible only when its functional character is understood. By overemphasis on manipulation, this is easily lost sight of. The variable x and the function $5x^2 - 2x + 4$ in which x occurs are to be thought of as variables rather than as numbers having but one or several fixed values. It is common experience that in analytic geometry two very important concepts, the variability of x and y as coördinates of a moving point and the conception of the equation of a curve as a relation between x and y are exceedingly difficult to the learner if in his previous training he has always considered x and y as unknowns having fixed values.

Functionality in Ratio, Proportion, and Variation. On account of the overemphasis on the logical organization of mathematical materials we frequently overlook the functional relations

in topics that are entirely relational in character. We have broken up "proportion" into the study of such isolated principles as alternation, composition, division, and inverse variation. It is customary to give these matters topical treatment, to study them intensely for a brief time, and then to drop them. Thus the pupil learns in geometry that the area of an equilateral triangle is found by means of the formula $A = \frac{a^2}{4}\sqrt{3}$ but remains ignorant of the fact that the area varies directly as the square of the side. The result is that he has not so broad an understanding of this formula as the pupil who knows that it expresses a relation between the side and the area of an equilateral triangle. Similarly, the relation between variation and proportion should be brought out clearly. It furnishes excellent opportunities for training in functional thinking.

The Graph Is a Means of Developing Functional Thinking. In the teaching of graphs, we often overlook the fact that making the graph is not an end in itself, that it is equally important that the pupil learn to "think graphically" of relationships. Many textbooks still present the graph as a separate topic and fail to make use of representation later in connection with other topics, even in cases where it might be a real help. Hence, graphic representation does not become a method which the pupil employs as effectively in dealing with quantitative relations as he does other mathematical methods.

From the beginning and throughout the course in mathematics the pupil should be taught to think graphically about numerical facts. His first experience will probably be the representation of numerical facts which do not involve any relationships. Such facts are usually represented by "bar graphs." One of the values of the bar graph is that it introduces the method of graphic representation in a simple manner. The relationships in bar graphs may be brought out by comparing the lengths of the various bars. Examples suitable for this type of graph are tables containing statistics about populations, areas of various countries, different crops of one country, exports and imports, and the like.

From the bar graph to the line graph is but a simple step. The line graph is especially helpful where changes are involved, as in records of temperature, growth, and population. Not only the making of the graph but the "interpretation" of the graph and the study of the relations involved are the ends to be attained.

Moreover, all precise mathematical laws that are stated as formulas and equations may be studied profitably by means of the graph. Thus the graph explains why the solution of a system of simultaneous equations is a *pair* of numbers, why a quadratic equation has *two* solutions, and why a cubic equation has *three.* From the graph of a trigonometric function the student sees at a glance all of its most important properties. Thus the graph shows that the sine function varies continuously from $+1$ to -1; that the sign is plus in the first two quadrants and minus in the third and fourth; that $\sin(180 - x) = \sin x$; and that after 360° all the values of $\sin x$ for $x = 0°$ to 360° are repeated. Indeed, trigonometric functions must be studied graphically if complete understanding is to be attained. Yet, in some textbooks on trigonometry, the graph is still inserted as a separate topic or chapter, somewhere near the end of the course with the implication that it may easily be omitted. Without the graph, the pupil will probably understand only one aspect of the trigonometric function, that is, the ratio aspect. He may miss the function idea entirely.

The algebraic procedure in *solving* simultaneous equations is highly abstract to the student and it introduces several elements that are puzzling to him. For example, the fact that a pair of equations, one of which is linear and the other quadratic, or two quadratic equations may have one or several solutions has but little meaning to the learner unless the various steps taken in solving are made concrete by a graphic presentation. If the graphic solution is taught first, the pupil will have little difficulty with the algebraic solution when it follows.

Demonstrative Geometry and Functional Thinking. In geometry so much attention is given to logical demonstrations that the study of functional relations is being greatly neglected. This is particularly unfortunate because most of the learning units of geometry deal with relationships. In the following it will be shown how functional thinking may play a larger part in the study of geometry.

The usual definitions of the concepts of geometry are only vaguely understood. Thus, when an angle is defined as a "figure formed by two intersecting lines," the learner sees only one aspect of the angle. In mechanics and astronomy angles are thought of as changeable figures, depending upon the amount of rotation of a line from one position to another. This is essential to a clear

understanding of what is meant by the *size* of an angle. For it is natural for the pupil to assume that longer sides make larger angles. The right angle, the straight angle, and the angle of 360° are merely positions of prominence as the angle changes from 0° to 360°. In all other positions of the rotating line the angles are acute, or obtuse, or reflex.

In many cases the relationship involved in a definition can be expressed in algebraic symbols. A few illustrations will suffice:

"Perpendicularity" means equal adjacent angles. It is expressed by means of the equation $a = b$, where a and b are the numerical measures of the two adjacent angles formed by two intersecting lines.

"Isosceles" means equality of sides, or $a = b$.

"Bisection" means equality of angles, or $m = m'$.

"Complementary angles" means that the sum of two angles is equal to 90°. The relation between the angles is $a + b = 90°$.

For "supplementary angles" the relation is $a + b = 180°$.

"Congruence" means equality of corresponding angles and sides.

"Similarity" means equality of angles and proportionality of sides.

For practical purposes the symbolic way of stating definitions is much more helpful to the pupil than the verbal definition. Thus, when a line is drawn bisecting the vertex angle of an isosceles triangle, the equality of the two parts may be used to establish the congruence of the two triangles formed. If the angles are marked m and m', this will suggest the equality $m = m'$, and the pupil will easily see that it is one of the facts to be used in proving the triangles congruent.

Relationships may be Shown by Making Changes in Figures. A number of theorems may sometimes be grouped together to illustrate the same general principle. Examples of this type are theorems dealing with measurement of angles by arcs, the theorem of Pythagoras, and its generalizations, theorems of proportionality of segments of chords and secants, and theorems stating proportionality of segments of transversals cut off by parallel lines.

In the first example the "general problem" is the measurement of an angle formed by two intersecting lines by means of the intercepted arcs of a circle which they touch or intersect. By changing the positions of the lines so that the point of intersection lies within, or on, or outside the circle, and by turning the sides of the angle

so that they become tangents or secants, the relationship between the specific theorems and the general principle is easily recognized. The pupil not only studies the separate cases but masters the general principle.

In the case of the theorem of Pythagoras the general problem is to show the relation between the sides of a triangle, or to express one side of a triangle in terms of the other two. By keeping two sides of the triangle fixed and changing the included angle from an acute angle to a right angle, and to an obtuse angle, the square on the side opposite the changing angle is at first less than, then equal to, and finally greater than the sum of the squares on the other two sides. Thus, what at first seem to be three distinct theorems are actually only specific cases of a general principle. In every case the teacher ought to emphasize this fact.

Relationships in Areas. Areas of triangles may be found by rules or formulas without calling attention to any functional relationships. However, the pupil will have a much broader view of the problem if he is made aware of the relations involved. Thus, some of the formulas show how the area depends on the base and altitude, or on two sides and the included angle, or on the three sides. Other formulas show that the area is a function of the radius of the inscribed circle or of the circumscribed circle. Which formula is to be selected for computing the area depends upon which parts are known or can be found by measurement. Furthermore, since the side of a regular inscribed polygon is a function of the radius of the circle, it follows that the area depends upon the radius. The study of areas of figures offers many opportunities for training in functional thinking.

From what has been said it is seen that much of the content of geometry involves dependence and correspondence. In fact, practically every unit of geometry deals with relationships. In parallel and perpendicular lines we have relations between angles and relations between the segments formed; in the circle relations exist between arcs, diameters, secants and chords; and in regular polygons we have the relations between perimeters, areas, and sides. Most of the topics that have not been mentioned, such as loci, concurrent lines, inequalities, are greatly simplified and clarified when the pupil's attention is called to the relations that are involved. If the teacher pays systematic attention to functional relations in all courses in mathematics the truth of Klein's state-

ment that the function is the soul of mathematics will take on greater significance.

What Teachers Think. Many teachers are skeptical about teaching the function concept in secondary school mathematics. Indeed, some insist that the topic belongs entirely to the higher courses. They underestimate the extent of experiences and abilities of their pupils. The ideas of correspondence, dependence, and relationship are very simple. They are encountered constantly in everyday life and fall within the experiences of every pupil. They can be understood without difficulty, especially if graphic representation has been taught.

Others agree as to the importance and desirability of developing functional thinking, but misunderstand the method of attaining it. It is not advisable to teach relationship as a "topic" of algebra or geometry. As in the case of all new concepts the idea of relationship is best developed when it grows out of the pupil's numerous concrete experiences with numerical relationships, and when such experiences are provided throughout the course, beginning in simple forms and as early as possible. Wherever relationships of quantity appear attention is to be drawn to them, and ultimately functional thinking becomes a method with the pupil. No definite place in the course can be named when the pupil should attain the ability to appreciate relationships whenever they occur. Ability to do functional thinking will be attained by individuals within time periods of varying length, but until it has been acquired the true purpose of mathematics has not been accomplished. Objective evidence will not be lacking when a pupil has grasped the idea of functionality. A few illustrations of such evidence will be sufficient.

Illustrations of the Adaptation of Functional Thinking. In studying the relation between the inscribed angle and the intercepted arc of a circle, a teacher was explaining to his class that as the angle changes from 0° to 180° the intercepted arc changes from 0° to 360°. At this point one of the pupils asked: "What is this relation like if represented graphically?" Later he himself made a graph picturing the relation. Evidently he had reached the stage where he was capable of functional thinking.

A difficult puzzle problem had been submitted for solution to the students of the University of Chicago High School and various solutions had been received.[1] It was found that a seventh grade

[1] Stone, Charles A., "Correlation of Mathematical Subjects Develops Mathematical Power," *Mathematics Teacher*, Vol. XVI, pp. 302-10 (1923).

pupil had obtained a correct solution by means of the graph. The pupil had not gone far enough in the course to attack problems by algebraic methods, but he had already reached the stage where he could visualize the facts of the problem sufficiently to represent them graphically. Ability to do functional thinking had come to him very early.

A third pupil was spending his vacation as a clerk in a hardware store. It was customary in this store to count out certain articles that were sold in large quantities. For example, large bolts that farmers bought were sold at a certain price for each. After a few experiences of counting, he began to think about the relationship between the number of articles and the price they sold for. It then occurred to him to weigh the bolts and to construct a graph that could be used to determine the price of any required number by weighing, which was quicker and easier than by counting. This pupil had mastered the function idea to the point that he was able to apply it as a method in his daily work.

Evidence of functional thinking is often found when a new topic is taken up for the first time. For example, in the beginning of trigonometry the pupils themselves frequently suggest the graphic representation of sin x as a method of representing the functional relationship between x and sin x and of studying the behavior of this new function. Thus the graph concretely illustrates that sin x is always positive in the first two quadrants and negative in the other two, that sin x can never exceed $+1$ and can never be less than -1, that an increase in x causes a corresponding increase in sin x but a decrease in cos x.

Summary. The following points summarize this discussion:

1. It has been shown that the function concept does not receive the deserved emphasis in secondary school mathematics. This may be accounted for in several ways:

a. Training in functional thinking is too often associated with algebra only; for example, with such topics as the formula or the graph, although most of the units in geometry deal with relationships.

b. Attempts to develop functional thinking by means of isolated chapters, or topics, are not successful.

c. There is a feeling among some that functional thinking is beyond the ability of the ordinary high school pupil.

2. A systematic study of the content of secondary school mathe-

matics reveals an abundance of opportunities for giving attention to functional relationships.

3. Evidence is obtained to show that high school pupils do acquire a grasp of the idea of functionality if it is taught throughout the course whenever relationships occur.

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DYNAMIC SYMMETRY

By MARIE GUGLE

Assistant Superintendent of Schools, Columbus, Ohio

Relation between Mathematics and Art. At first thought, there seems to be little relation between mathematics and art. The one deals with abstract thinking which is mostly symbolic; the other portrays beauty and makes an emotional appeal. Nevertheless, the recent rediscovery of the geometric bases of classic art of the fifth and sixth centuries B.C. and their subtle proportions may be used effectively by teachers of mathematics to give new meaning and new significance to such topics as square root, surds, reciprocals, and the golden section, topics usually treated in a formal way. It is the purpose of this chapter to discuss some of these important relationships.

Static and Dynamic Symmetry. Since the days of classic art, artists have used what is called static symmetry. The dimensions of its designs are commensurable. Their lengths and widths have definite arithmetic measures. The geometry of static design consists of regular geometric figures, perhaps superposed at different angles, as two squares put one on the other at an angle of 45°. The art of the classic periods in Egypt and Greece used a more subtle proportion, as was recently rediscovered by the late Jay Hambidge of Yale University. He gave it the name of dynamic symmetry. In designs of this proportion, the dimensions are incommensurable. The most common ratios are $1 : \sqrt{2}$, and $1 : \sqrt{3}$, and $1 : \sqrt{5}$. Dynamic designs are evolved by the use of areas rather than by line measurements. They are based on root rectangles which have the foregoing ratios.

Problem I

To Draw Root Rectangles

(a) Draw a square and its diagonal. Use the side of the square and its diagonal as the dimensions of a rectangle. The result is a *root-two rectangle.*

(b) Use the side of the square and the diagonal of the root-two rectangle as dimensions of a *root-three rectangle.*

(c) Continue the process to produce a *root-four rectangle* (which is a double square) and a *root-five rectangle.*

In dynamic symmetry there are two other basic figures. One is the reciprocal rectangle and the other is the whirling square. We shall now consider the former.

Problem II

To Draw a Reciprocal Rectangle

(a) Draw a root-two rectangle and its diagonal. From one of the opposite vertices draw a line perpendicular to the diagonal and produce it to meet the opposite side. The new rectangle is a reciprocal root-two rectangle. Its area is

$$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = \frac{1}{2} \text{ of } 1.414 = 0.707$$

(b) In a similar manner draw a reciprocal root-three rectangle.

$$\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = \frac{1}{3} \text{ of } 1.732 = 0.577$$

(c) In a similar manner draw a reciprocal root-five rectangle.

$$\frac{1}{\sqrt{5}} = \frac{1}{5}\sqrt{5} = \frac{1}{5} \text{ of } 2.236 = 0.447$$

Problem III

To Draw Root-Rectangles in a Quadrant

(a) Draw a square $ABCD$ and the diagonal AC.

(b) With A as a center and AB as a radius draw the quadrant DB cutting AC at E.

(c) Through E draw the line FG parallel to AB. Then $ABGF$ is a root-two rectangle in the square AC.

(d) Draw the diagonal AG, cutting the quadrant at H.

(e) Through H draw the line IK parallel to AB. Then $ABKI$ is a root-three rectangle in the square AC.

(f) By continuing the drawing of diagonals and parallels, root-four and root-five rectangles may be constructed.

Problem IV

To Draw Design Patterns from Root-Two Rectangles

(a) Draw a square *ABCD* and a root-two rectangle on the side *AB*.

(b) In like manner draw a root-two rectangle on *DC*; on *AD*; on *BC*.

(c) These overlapping root-two rectangles divide the square into five smaller squares and four root-two rectangles.

By drawing various diagonals in the smaller parts an almost infinite variety of design units can be evolved. Different designs may be worked out in patterns for silks or book covers.

Symmetry in Nature. Even more interesting than their use in design is the use of root rectangles, diagonals, and reciprocals in nature. Dynamic symmetry is the key to the maple leaf; the plan of the dragon fly is based on the root-three rectangle. While we do not see much resemblance between the dragon fly and the iris, yet both are made on the same geometric pattern. The side view of the iris fits into the dynamic root-three pattern. Such a view shows the growing plant, and the word dynamic means growth or power. It is interesting to note that when the flower comes to fruition, the top view fits into a regular equilateral triangle with a smaller one within, which, of course, is a static symmetry pattern. The construction of this static pattern is made, not by drawing two triangles but by drawing the root-three rectangles, its diagonals and perpendiculars, extending them to make the sides of the outer triangle, and connecting the two poles and the mid-point of the base to form the inner triangle. Static symmetry, therefore, is the outgrowth and completion of the dynamic symmetry, just as the bloom is the completion of the growing plant.

Problem V

To Draw a Root-Five Rectangle in a Square

(a) Draw a square *ABCD*.

(b) Within the square, draw a semicircle on the side *AB*, whose mid-point is *O*.

(c) Draw *OD* and *OC*, i.e., the diagonal of half of the square. These lines cut the semicircle at *E* and *F*.

(d) Through *E* and *F* draw *GH*.

(e) $ABHG$ is a root-five rectangle.

(f) If the side HB is unity, then AB is $\sqrt{5}$ or 2.236 and EF as well as $E'F'$, its projection on AB, is equal to 1.

Then, $AE' = F'B = \frac{1}{2}(2.236 - 1) = 0.618$

And, $AF' = E'B = 1.618.$

In the root-five rectangle the figure AF or $E'H$ is a whirling square. Its ratio is 1.618, which is the reciprocal of 0.618; for $1/0.618 = 1.618$. This ratio, 0.618, will be recognized as that resulting from dividing a unit in extreme and mean ratio, or the golden section of the Greeks. A line is divided in extreme and mean ratio if the whole line is to the longer segment as the longer is to the shorter segment. This is a theorem which should be familiar to every high school teacher of geometry.

Problem VI

To Draw a Whirling Square

(a) Draw a square $ABCD$.

(b) From O, the mid-point of AB, draw OC.

(c) With O as a center and OC as a radius, draw the arc CE, meeting AB produced at E.

(d) On AD and AE complete the rectangle $AEFD$, which is a whirling square rectangle, whose width is 1 and whose length is 1.618.

Problem VII

To Draw a Root-Five Rectangle from a Whirling Square

(a) Use the figure of Problem VI.

(b) From O, draw the arc DG and complete the rectangle on the opposite end in a similar manner. Then the whole figure $GEFH$ is a root-five rectangle. For,

$$AB = 1.000$$
$$GA = 0.618$$
$$BE = 0.618$$

Therefore, $GE = 2.236 = \sqrt{5}$

Because the root-five and whirling square rectangles are so intimately related, these two types may be combined in the same design.

Problem VIII

To Show the Reason for the Name Whirling Square Rectangle

(a) Draw a square *ABCD* and on it construct the whirling square rectangle *AEFD.*

(b) Draw the diagonal *AF,* which cuts *BC* at *G.*

(c) Through *G,* draw *GH* parallel to *BE.*

(d) Then *BEHG* is a square.

(e) Within the rectangle *HC,* draw on *HF* the square *HFIJ.*

(f) Continue to draw squares on the end of each remaining rectangle. These squares form a series of squares which whirl around a point or pole on the diagonal.

(g) This process may be reversed thus: Start with a small whirling square; then draw an external square on the length of the original whirling square, and then another square on the length of the new rectangle. Each new square makes an enlarged whirling square rectangle out of the whole figure.

Problem IX

To Draw a Reciprocal Whirling Square Rectangle

(a) Draw any whirling square rectangle and its diagonal.

(b) From one of the other vertices draw a perpendicular to the diagonal, produce it to meet the opposite side, and complete the rectangle. The new one is the reciprocal of the original. The intersection of the diagonal and perpendicular is an eye of the whirling square.

Since each of the two diagonals may have two perpendiculars, there are four possible eyes in a whirling square rectangle. By drawing parallels through these eyes, we get a pattern which offers the possibility of an almost infinite number of variable designs.

Of these variations of the whirling square, only one can be described, that of the 1.382 shape, which, it is found, was much used in both Egyptian and Greek art.

Problem X

To Construct a 1.382 Shape in a Whirling Square

(a) Draw a whirling square *ABCD.*

(b) Draw its two diagonals, and, from the opposite vertices, draw two perpendiculars to each diagonal.

(c) Through each pair of eyes, draw two lines GE and FK, which are parallel to the lines AD and BC.

(d) Letter the two upper eyes I and H, respectively. Draw the line IH.

(e) In GH, draw a square on GI, leaving a whirling square.

$$EH = \text{a square} = 1$$

$$\square\ GH = \text{a reciprocal } S + WS$$

$$= \frac{1}{1 + 1.618} = \frac{1}{2.618} = 0.382$$

$$\square\ EF = \square\ EH + \square\ GH$$

$$= \quad 1 + \quad 0.382$$

$$\text{Therefore, } EF = 1.382$$

The shape discussed above is fundamental in many Greek vases.

Uses of Symmetry. In his book *The Greek Vase,* Mr. Hambidge has analyzed over two hundred Greek vases and fitted each into its own dynamic symmetry pattern. The pattern of the Nolan amphora from the Fogg Museum of Boston is based on a 1.7071 shape, which is a square and a reciprocal root-two rectangle. The beautiful Kantharos of the fifth century B.C., now in the Boston Museum of Fine Arts, has a dynamic symmetry pattern based on a compound shape derived from a root-five rectangle.

There is a unique connection between the whirling square and the plan of leaf-distribution in plant growth. This distribution is based on the Fibonnaci or summation series, whose ratios approach 1.618, the ratio of the whirling square. Another fascinating phase of this study is found in connection with the pentagon, which is closely related to the leaf, to the summation series ratio, and to the whirling square.

Egyptian bas reliefs have been analyzed into their dynamic symmetry patterns. The Greeks used these subtle ratios in the architecture of their temples as well as in the designs of their vases. For example, the long side of the Temple of Apollo at Bassæ, in Arcadia, has a ratio to the end of four times 0.618. To lay out such a temple is an interesting feat. First an orientating line, as AB, is defined, probably by sighting along a stick. The main axis of this building is from north to south, an unusual orientation for a Greek temple. On AB as the long side, a double square, $ABCD$, is constructed, determined by the prescribed width of the temple as a side. A man standing at A would hold one end of a

rope, the other end of which would be carried to C by another. The man at C would walk to E, in line with AB, drive a stake and return to C. Then from A, the first man would walk to G, in line with CD, and drive a stake. Then the rectangle, $EFGH$, is easily completed. This rectangle, the outer one of the temple, is four times 0.618, shown as follows:

$$\begin{aligned}
AF &= HC = \text{root-5 rectangle} \\
HB &= AE = \sqrt{5} = 2.236 \\
HA &= BE = 2.236 - 2 = 0.236 \\
HE &= HB + BE \\
&= 2.236 + 0.236 \\
&= 2.472 \\
&= 4 \times 0.618
\end{aligned}$$

The temple has two inner courts, each determined by intersections of lines. The second court, the naos, has two 1.618 areas; that is, each is a whirling square. The inmost court, the cella, has an area similar to the outer rectangle; that is, it has four times a 0.618 area.

We find that dynamic symmetry was used by the ancient Greeks and Egyptians, and then in the Middle Ages artists dropped back to static symmetry. The Creator himself used dynamic symmetry in his creation of plants and animals, for the butterfly fits into a root-five and whirling square design. Even the human skeleton has been found to have the proportions of dynamic symmetry. The Greeks probably discovered their ratios through a study of nature and then applied them to their own temples and vases. The Greeks, overcoming their lack of instruments, could use the proportions of dynamic symmetry because they are commensurable by areas.

What Modern Artists Have Done. Within the last few years modern artists, as the late George Bellows, Howard Giles, and Leon Kroll, have revived these ratios by using them in their drawings and paintings. Every modern art school is teaching this symmetry. No doubt, through a study of Hambidge's rediscovery of the principles underlying the subtle beauty and satisfying proportions of Greek art, modern artists, including painters, sculptors, potters, and architects, will help to create a new era in art. It is hoped that the jazz art of today will soon be replaced by a real art, whose lasting beauty depends upon the subtile use of number.

Some people fear that such mathematical precision may hamper the artist. According to Miss Gisela A. M. Richter of the Metro-

politan Museum, this is not the case. Concerning the Greek vases as analyzed by Hambidge and others, she says:

> Each of these works of art forms a composition, in which the various details repeat the primary theme, thus producing a unified harmony. That is, the rectangles made by the lip, the neck, the body, the foot, and the handles, respectively, bear a definite mathematical relation to the rectangle which contains the vase as a whole.
>
> We have an interplay of areas comparable to the sequence of phrases in a musical composition. It no more impedes the artist than harmony obstructs the composer or a metric system the poet. It merely supplies the law and order, which we know to be one of the dominant characteristics of Greek art.

Place of Symmetry in Teaching Mathematics. Teachers of mathematics will find a fascinating study in Hambidge's symmetry. Without elaborate illustrations, we are able here to give only a suggestion of a few of its many phases, interesting in themselves and as valuable applications of square root and other topics in mathematics. It has been found that eighth grade pupils not only understand root-rectangles but are eager to make the drawings which vitalize their work in square root. For senior high pupils the study of the pentagon, surds, and extreme and mean ratio will take on an entirely new significance through Hambidge's discovery. Furthermore, dynamic symmetry marvelously increases for both teacher and pupils appreciation of mathematics and its power.

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INTRODUCTORY CALCULUS AS A HIGH SCHOOL SUBJECT

By M. A. NORDGAARD
St. Olaf College, Northfield, Minnesota

I. Introduction

Significant Sidelights from the History of Mathematics. In the kingdom of knowledge it is youth that inherits the land. The third century B.C. reached in Euclid's *Elements* the high-water mark in Greek mathematics. This fountain of learning, from which drank the coming centuries, was for philosophers and scholars, sages and savants, and not for the young men of Athens, Alexandria, Syracuse, and Samos. Yet now for two hundred years young people in their middle teens have had their wits stimulated and their logic straightened by this same Euclid. The trigonometry of Aristarchus and Hipparchus was once appreciated only by the Theons and the Ptolemys, the Peurbachs and the Keplers; but long since has it ceased being the monopoly of astronomers scanning the heavens, and is now to be found in the secondary schools of all countries. The Hindu method of extracting square and cube roots developed in the sixth and seventh centuries, the Hindu-Arabic notation introduced into Europe in the tenth century, the method of long division evolved in the fourteenth century have been taught successively in the universities, the secondary schools, and now in the elementary schools. Logarithms, that handmaid of science invented three hundred years ago, was for decades considered suitable only for college and university students; yet it, too, was taken up by the secondary schools, and today we see even youngsters in their early teens calculating with logarithm tables and manipulating the slide rule.

And so nearly all the important mathematical inventions and discoveries of the race have found their way from academy porticoes and university halls to the classrooms of the schools. The subject matter has been adapted to the new audiences, some portions of it have been omitted and others enlarged upon, new methods

of attack and new avenues of approach may have been found advisable; but almost every mathematical field studied by the race has been found valuable in the elementary and secondary schools as equipment for practical life or for stimulating energy and giving intellectual pleasure.

Analytic Geometry and the Calculus. Strangely enough, the two most important mathematical contributions of the seventeenth century have been very slow in coming down to the secondary schools. They seem to have held out against the general law like strongholds in a belated feudalism. At least this is true in the United States. I refer to analytic geometry and the infinitesimal calculus. The first of these citadels has now fallen. Not as the analytic geometry of our college course do we find this subject given; but in most high school algebras now in use, and in many arithmetics, we find the function concept employed and graphic methods applied to problems. This transformation has taken place nearly altogether in the twentieth century. The graph, the formula, and the algebraic equation functionally interpreted are a schoolbook product of our day.

The calculus still holds out. Not in France, Germany, Austria, England, Russia, Italy, Spain, Norway, Sweden, Denmark, Holland, and Belgium, where it has been incorporated into the secondary school program for some years,—but in the United States. Yet the castle has been stormed,—some of the outer bulwarks are even down. Mentors of American youth have begun to see the value of the calculus in a liberal education and mathematics teachers are beginning to have crystallized views as to its place in the high school program. A great step was taken in American mathematical pedagogy when the National Committee on Mathematical Requirements in their epochal Report on the Reorganization of Mathematics in Secondary Education came out with the recommendation that "elementary calculus" be offered as an elective in qualified senior high schools.

A few schools and individual teachers had already taught the calculus to high school classes before the report appeared and other schools have since taken it up. A beginning has been made. It is safe to say that as the Euclid of the third century B.C., as the Hindu-Arabic notation and the Arabic algorism of the Middle Ages, as the algebra developed during the Renaissance have long been taken up by the lower schools, and as the logarithms and

coördinate geometry of the seventeenth century have in our own day found a place in the high school algebras, so will the infinitesimal calculus, the crowning mathematical achievement of the seventeenth century, be adopted as a high school subject.

Purpose of This Chapter. We shall first try to analyze the mathematical situation and the pedagogic temper that has at this time led to a call for teaching the elementary calculus in the high schools of our country. Next, we shall trace the stages by which this subject has found its way into the curricula of the secondary schools in the leading countries of Europe. We shall then give a description of the experiments that have been carried on in this field in the United States in recent years. And lastly, we shall offer a few specific suggestions as to how to open up the subject in a pre-college course.

II. An Earlier Place for the Calculus in the Curriculum

Tendency of Subjects to Move Downward. All mathematical disciplines have shown a tendency to move downward as they have become more widely known.

While a new field of mathematics is still in the process of development, most people can have little or no part in it. New facts and new theories are contributed by a select few. Then there comes a time when one or more master minds will build this promiscuous knowledge into a body of truth which the scientific world will evaluate and utilize. As the new science forms more contacts with life, it enters deeper into people's thoughts, and an acquaintance with it will be considered part of a liberal education. As the general level of education is raised, the younger strata of population become capable of receiving benefit from this study. Youth will want his heritage, and the subject will be taught in the secondary, perhaps in the elementary, schools. That has been the history of arithmetic, geometry, trigonometry, algebra, and analytics.

The Calculus. In its origin the calculus was a study of functions invented by, suited for, and utilized by men who did research in mechanics, optics, and advanced mathematics. This science, invented independently by Newton and Leibniz in the latter half of the seventeenth century, needed a hundred and fifty years of amplifying, systematizing, and refining by such men as Taylor, Maclaurin, Euler, Lagrange, and Cauchy before it could be em-

ployed with ease and effectiveness by scholars not specializing in mathematics. In Europe it was first taught only in the universities. In America it has been confined to institutions of college grade. When Harvard first found a place for it, she placed it in the upper classes, and this plan was followed by other colleges that offered the course. But it has gradually worked its way downward. For this the engineering colleges have been largely responsible, the calculus being used mainly as a foundation for the technical courses. For years its place was stabilized in the sophomore year. But recently many colleges have placed the simpler portions of the calculus in the freshman year, frequently as part of a unified course. As to the wisdom of this latter arrangement there is at present no agreement. Some colleges have discontinued the plan, but more are taking it up. There is no mistaking, however, which way the wind blows. The calculus is moving downward.

Two Main Reasons for the Change. The most patent, though not the most fundamental, reason for the change is the age in which we live. Thanks to our generous system of high schools and modern means of communication the intellectual interests of the rank and file have never been so numerous nor has their participation in the conduct of the world's affairs ever been so active as at the present time. Ours is a scientific age, and science speaks through mathematics, notably through the calculus. Not only do physics, mechanics, and engineering depend on the calculus—everyone knows that—but radio-activity, medicine, and actuarial science frequently state the results of their latest researches in calculus form. Even insurance, statistics, heredity, and education speak the language of the calculus. A recent elementary textbook on wireless telegraphy uses some calculus. In these days of ocean flights the aviation journals frequently contain articles that cannot be understood without at least some knowledge of this subject. To know some calculus has become a necessity for an educated person.

While teaching at Columbia University the writer had in his class a student working as assistant at the Rockefeller Institute. In his work this student ran across graphs and equations dealing with the healing of wounds and the growth of tissues, where the calculus and other mathematics were means of expression as well as tools of work. This student had a fair degree of education; yet these operations were meaningless to him. He took courses in mathematics in order to get enough knowledge of it to appreciate

what he saw done in his working environment. But that is uphill work for a man working full time. I mention this incident in order to lend point to the following quotation from the Report of the National Committee on Mathematical Requirements (page 34):

> If the student who omits the mathematical courses has need of them later, it is almost invariably more difficult, and it is frequently impossible, for him to obtain the training in which he is deficient. In the case of a considerable number of subjects a proper amount of reading in spare hours at a mature age will ordinarily furnish him the approximate equivalent to that which he would have obtained in the way of information in a high school course in the same subject. It is not, however, possible to make up deficiencies in mathematical training in so simple a fashion. It requires systematic work under a competent teacher to master properly the technique of the subject, and any break in the continuity of the work is a handicap for which increased maturity rarely compensates. Moreover, when the individual discovers his need of further mathematical training it is usually difficult for him to take the time from his other activities for systematic work in elementary mathematics.

In educational circles the calculus was, until recently, thought of only as a professional course in schools of technology or as a disciplinary subject in liberal arts colleges. Its cultural value in interpreting our environment and its direct usefulness in practical work have not been appreciated until the last few years. It is this changing attitude on the part of educated people that brings up the question of putting the calculus into the high school curriculum.

The second reason for the on-coming change is a mathematical-pedagogical one, and is certainly the most fundamental. Whatever other reasons there may be for including the calculus in the high school program they would not sufficiently justify the step unless they were in harmony with the central idea of mathematical curriculum making that has been crystallizing for the last fifty years and has come directly to the fore during the last three decades. The last twenty-five years have been a period of storm and stress in the guild of teachers of mathematics. External conditions have been full of changes, an unprecedented number of people have sought a higher education, the secondary schools have almost become "common schools," and these larger groups clamor to taste of the knowledge that was formerly reserved for the few. Consequently existing forms of secondary education are being examined and recast in the light of modern psychology and in order to meet

the new demands. New principles of instruction have been enunciated, new methods devised, and new content inserted into the curriculum.

The Function Concept. The essence of modern reform is that the *function concept* shall be the central idea in *all* mathematical teaching. And for this there are two reasons. First, the student will learn more actual mathematics. By organizing the course about a fundamental principle like the function concept there is prevented much waste of time and effort in dealing with unrelated themes and the student will therefore get *further into* mathematics. The important principles and topics also will be kept working continuously, thus making for thoroughness. Second, the student will be in a better position to apply his mathematics to his environment and to his future studies. Learning to look for connections that exist between related quantities will enable him to think of quantities with which he will have to deal in real life whether he takes any more mathematics or not. For in the material world about him the actual existence of one variable quantity depending on another is a commonplace in the student's experience.

America, with her wide spread of territory and local direction of schools, has been very slow in adopting this reform. But in Europe the new practice is well established. In France and Germany functional thinking is inculcated in the early years by means of a number of concrete instances. In the sixth and seventh years the idea of abstract relationship is brought out by graphs and formulas. The children thus learn to recognize variable quantities and by concrete examples learn to think of x and y not only as unknown quantities to be determined but as changing quantities whose variation is to be studied. That is, the principles, but not the mechanism, of analytic geometry are introduced early in the pupil's career and are not left to be studied as a special topic at the end of the course or deferred till he takes up higher mathematics.

In the decade from 1880 to 1890, many of the secondary schools in Continental Europe took up the study of functions as a part of their program. From analytic geometry they adopted the graph for their work in algebra. Most schools have since adopted graphic algebra, whether they teach the function concept or not. In the United States graphic algebra has been quite general since 1910.

Until the creation of the National Committee on Mathematical

Requirements in 1916 the function concept had a very small place in the scheme of American education. Arithmetic texts had formulas and algebra texts provided work with graphs. But the idea of dependence was not stressed, and the teachers themselves did not utilize even the meager material furnished by the text. Not until they took up trigonometry and analytics, if they got that far, did students get systematic training in functional thinking. The influence of the Mathematical Association of America, which has always taken a keen interest in secondary school mathematics, and the aggressive work of the National Committee have done much to make known the functional idea among teachers of mathematics.

Misuse of Graphs. The graph is a pictorial representation of an algebraic law. The main purpose of the graph in algebra is to teach in a simple manner the idea of functionality. By it continuity and irregularity are made very plain. It lends itself especially to work with formulas. But in its early stages it was deflected from this useful course before very long, due to the very close connection between algebra taken up this way and analytic geometry. Both now dealt with functions and, before we were aware, algebra concerned itself chiefly with the curve itself, in the manner of analytic geometry, instead of with what the curve stood for,—with the properties of the curve instead of with the quantitative relationships, with the static elements of analytics instead of with the dynamic element of the algebraic problem. The function concept itself was lost sight of and much graphical drudgery, such as we see in many American algebras of today, took the time from more valuable pursuits. And so this latter-day attempt at enriching the elementary mathematics by materials and ideas from the higher mathematics was nearly frustrated.

Fortunately this futile work in graphs was abandoned. Pressure was brought upon those who were responsible for the mathematics of the secondary school to justify its presence in the curriculum. This caused the teachers of mathematics to examine their tenets in the light of experience. As a result a new type of graphic problems has found its way into algebra, especially in the early part of the course. Problems from science, statistics, and everyday life are studied, both those that have mathematical formulas and those that do not, and the functional aspects are emphasized. The Sunday paper supplement, the scientific journal, the popular magazine, and the advertising circulars testify to what large extent the

graph and its companions, the formula and the equation, have become a part of our daily lives. The reading world is becoming functionally-minded.

An idea cannot have a flourishing growth unless it has symbols of expression and tools to work with. It is doubtful if Newton and Leibniz would have gotten far into the infinitesimal calculus without Descartes' coördinate geometry. That vivid portrayal of functional relations gave wings to the calculus. And so it has been among humbler folk than Newton and Leibniz. It is the popularization of the graph that in these last years has made us functionally-minded, so that even in some railroad offices charts will tell exactly where a certain train is to be at any given instant.

At first the algebras and arithmetics discussed only the general relation between the variables exhibited by the graph. In some instances the method of interpolating values was taken up. But the graph was seen easily to bring out other truths, such as maximum and minimum values, areas, and rates, both average and instantaneous. This latter especially necessitated the notion of limits, a notion not hitherto included in the algebras. That led up to the threshold of the calculus, and the very suggestion threw open a vast field of unused possibilities.

It is these unused possibilities, so convincingly set forth by Professors Klein of Germany, Perry of England, and D. E. Smith of the United States, that have led to the adoption of the calculus in the secondary schools of Europe and have stimulated the movement to offer it in American high schools.

III. Practical Questions to be Considered

Calculus in the High School. Let the question of teaching the calculus in the high school be raised, and in the mind of some teacher is conjured up a picture of the calculus class in the sophomore or perhaps the junior year of college, where a learned professor expounded to the intellectual élite of the college the Mean Value Theorem, singular points, or moment of inertia. Such a teacher looks at the seventeen-year-old boys and girls of his high school and cannot imagine these youngsters taking the calculus. But if he will also remember the ellipses and hyperbolas in his own study of analytic geometry, and then glance at the subject matter in the high school algebra which he teaches, he will become more receptive to this other suggestion.

For those who urge this innovation do not advocate taking up the more difficult parts of the calculus. They recommend that only such topics and methods be considered as are within the comprehension of the students and which are fertile in illustrating the principal concepts of the calculus. As the most fitting topic for such a purpose the National Committee recommends the algebraic polynomial. Quoting from the Report:

> The calculus of the algebraic polynomial is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, and of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations—and such training is of value to everyone.

Some authorities hold that the manipulative work should not be onerous, that computational dexterity is not the purpose of the course, that discussion should be adequate, but not necessarily complete. One says, "State nothing but the truth, but not necessarily all the truth." Still others will maintain that in the beginning, mathematical intuition should be allowed to do its work; too fine rigor is not in place till later. Says an English schoolman on this point: [1] "For more than a hundred years after the formal development of the calculus the most eminent mathematicians of Europe ignored these difficulties with which the rigorist wants to perplex beginners. We, as teachers, are just learning that you cannot teach people to generalize by throwing ready-made generalizations at them—that a grip on the concrete facts must precede a critical analysis of them."

The Practical Schoolman's View. But other points besides content and method need to be considered. To the practical schoolman, who is responsible for the organization of the school and the coördination of courses, four questions especially present themselves: (1) Must essential and vital parts of the general curriculum be sacrificed in order to make room for this new course? (2) Does the teaching force, as regards number and academic preparation, permit it? (3) Will it occasion a re-arrangement of the present mathematical curriculum? (4) Has a high school junior

[1] C. S. Jackson in "The Calculus as an Item in School Mathematics." *The Mathematical Gazette*, Vol. VII, p. 197.

or senior the mentality and experience to profit by such a course? Other questions may be asked but these four are vital.

Valid questions all, and necessary. As to the first two: The National Committee has recommended that elementary calculus be offered as an elective in the senior high school. The mathematics in the eleventh and twelfth grades has been elective for years, so no change is contemplated in this respect. With the rapid growth of the junior high school, there will eventually be a three-year required course consisting of the mathematics that in the minds of schoolmen should be known by every educated person. The mathematics in the senior high school will be elective; it has already been elective for years in the last two years of the four-year high school. Being elective, it will not crowd out any of the non-mathematical parts of the curriculum, as far as the students are concerned. But it will give the student with mathematical interest an opportunity to take up this interesting branch. And this is especially valuable to the person who will not go to college. As to the instruction, nothing could be more disastrous to a new course than to have it conducted by incompetent teachers. However, most schools now have teachers who have taken the calculus in college, and these can safely undertake to teach the elements of it to high school pupils. Nor should teachers out of sympathy with the new course be pressed to give it, no matter how thorough their education. If the teaching staff is small, it may be inadvisable to give the course. Yet often in such cases a course in the elementary calculus may be given as part of the higher algebra course, or it may alternate by years with some other elective mathematics.

As to the third point: With sufficient teaching force it will not be necessary to disturb the existing mathematical courses; one may simply add this extra course for high school juniors or seniors. But since we are now in the midst of a country-wide reorganization of our secondary school mathematics anyway, it will be well to have the calculus course in mind in that reorganization. In fact, the demand for a high school course in the calculus has sprung from the same movement that prompted this reorganization.

The usual course in mathematics will certainly continue to be elective in the last two of the senior high school years as it has been in the traditional high school. The feeling is gaining ground that with the improved course in the junior high school so much

demonstrative geometry has already been covered that an extra year in the senior high school should suffice for both plane and solid geometry. That leaves two years in which students with mathematical interests may select from one to four semester subjects. Thus far these have been higher algebra, solid geometry, trigonometry, and, in a few schools, an extra semester of algebra. Other studies that have been advocated are elementary statistics, mathematics of finance, surveying, and shop mathematics. One may safely predict that trigonometry and higher algebra will continue to appeal to high school students. But there is still time for at least one semester of elementary calculus.

One cannot help feeling, however, that parcelling out a semester to each of these is a very arbitrary arrangment. A semester of trigonometry five times a week takes a high school student further into the subject than the trigonometry work given by colleges, and certainly further than the educational environment of a high school student warrants. In the higher algebra courses so much material has been introduced from the theory of equations and the function theory that frequently the course passes beyond the intellectual needs of even the abler students and they find the subject dull and uninteresting. It would harmonize better with the student's capacity, adapt itself better to his experience, and form a better basis for his courses in science or later work in mathematics if, instead of going so far into higher algebra and trigonometry at this time, we let him use part of the year in learning the fundamental notions of the calculus.

In regard to the fourth question, has a high school junior or senior the capacity and experience to profit by such a course? In the first place, we want to dispel the idea that an introductory course in the calculus is harder than many topics now given in the high school course. There is no part of the contemplated work in elementary calculus so difficult as DeMoivre's theorem, the roots of unity, Horner's method, and series.

What do we find in other countries? In Austria, Germany, France, Hungary, and Sweden, certain schools introduce analytic geometry in the eleventh year, and in certain of the *Realschulen* of Austria and the *Oberrealschulen* of Germany the method of the differential calculus is developed and applied to physics in that year. In the twelfth year the schools of Denmark, Germany, Austria, Belgium, Sweden, Switzerland, Rumania, France, and Russia, and

some of the schools in England, offer the differential and integral calculus.

No one seriously believes that the mind of American youth cannot measure up to that of his European cousin. His school year may be shorter and his curriculum poorly planned. His teachers, too, may be less well equipped. But he certainly has the mental power to do in the eleventh and twelfth years what the European youth does in those same years. True enough, thanks to better trained teachers and a well planned curriculum the European student in his twelfth year is on a par with the American college sophomore in respect to scholastic attainments, but that does not affect his native capacity. And it should be a challenge to American schools to improve their teaching staffs and reorganize their courses in all departments so that the American high school student may have an even chance. *In this movement, the mathematics teachers should be in the van!* At the present time they are not taking advantage of their opportunities.

Someone may say that students in European secondary schools are highly selected, whereas with us the high schools are a cross section of the general population. However, the calculus given in French and German schools is required of all pupils in the schools. The calculus course we suggest is elective, and anyone who has taught elective courses in advanced mathematics in high school knows that they attract a superior class of students. No group of European secondary school students will be more select, at least as far as mathematics is concerned.

IV. Historical: The Introduction of Secondary School Calculus

In France. We may get some light on the movement which we are sponsoring by knowing something of its inception and early development.

In France the secondary schools have been giving work in the calculus for over a hundred and fifty years under *algèbre* and *analyse*. France has long been committed to the plan of bifurcating the courses—that is, giving a classical and a scientific course—in the upper years of her secondary schools, and she has encouraged a degree of specialization in the twelfth year (philosophy or mathematics form) that in most countries is permitted only in the university. The very exacting mathematical requirements for entrance

to the *école polytechnique* and the scientific department of the *école normale* have had a potent influence in giving direction to the mathematical work of the secondary schools. Since the time of Napoleon, school authority has been highly centralized and is directed from Paris. That possibly accounts for France being the first country in the world to include work in the calculus as a regular and required part of the curriculum in her secondary schools.

The modern French secondary schools date from 1802 when thirty *lycées* were substituted for the then existing central schools. By a decree of that year Napoleon directed that the subjects of the new *lycée* curriculum should be essentially Latin and mathematics. At the end of the regular program there was an additional two-year course in mathematics, known as *mathématique transcendante,* in the first year of which the application of the differential calculus to mechanics and the theory of fluids was discussed. This extra course was the precursor of the advanced mathematics classes that we find in the secondary schools of France today. We in America should know more about what the French have done.

A definite bifurcation of the course into the classical and the scientific wings took place in 1852. There has been considerable modification of contents and time schedule since then, notably in 1902 and 1905, but the principle of bifurcation and specialization has been retained.

In the last quarter of the nineteenth century there grew a movement that in due time should result in the calculus being taken up by all students in the *lycées.* A lively interest in the pedagogy of mathematics, stimulated by psychological study and responsiveness to the actualities of practical life, began to assert itself at this time. Correlation of mathematics with the sciences, the unification of the different branches of mathematics, and a psychological rather than a logical approach to the different topics were being agitated.

Above all the function concept was to be made the core and center in mathematical instruction. It is from this latter idea that the question of taking up the calculus by the non-specializing, non-preprofessional secondary school had its beginning, in France as well as in the other countries. Having a central idea like the function concept going through the entire course would strip mathematics of the numerous unessentials that had been grafted on to it, take the pupils further into mathematical theory, and give

more opportunity for continual applications. In 1898 Laisant wrote: [2]

> When the elements of arithmetic, of algebra, and of geometry shall have been freed from the multitude of parasitic propositions and reduced to the exposition of directive ideas and essential methods, not only will valuable time have been gained, but also greater clearness of ideas imparted. This will permit the introduction of the elements of analytic geometry and of calculus.

It was fortunate for France that preprofessional requirements and the special mathematics forms had demonstrated that the calculus *could* be mastered by students of secondary school age and attainments.

The discussion of the introduction of these advanced branches in the general course of study continued, linked up with the questions of the function concept, correlation, and method. In scientific and pedagogic matters the French have been a clear-thinking people, and ideas soon began to take form. To their leaders it seemed that the logical corollary to the adoption of the function concept was the introduction of the calculus. In 1902 the new French curriculum, effective throughout all France, extended the minimum of mathematics required of *all* students in the *lycées* to include analytic geometry and the calculus.

Since French schools teach all their mathematics as a unified course, they found an immediate use for the new study especially in their work with algebra, and their work on the calculus is now incorporated in the algebra texts. Even in branches that do not seem to have a close union with the calculus, as solid geometry, they found it useful. Thus J. Tannery in an article in the *Revue Pedagogique* for July, 1903,[3] urges that the equivalence of the volumes of oblique and right prisms be proved by integral calculus. To quote:

> Integral calculus! In the secondary school!! Yes, I am not joking. The effort needed to learn what a derivative is, an integral, and how by means of these admirable tools surfaces and volumes can be evaluated, is certainly less than the effort hereto demanded of a child to establish the equivalence of oblique and right prisms, of two pyramids (the staircase figure, you know, that is so hard to make), then the insupportable volumes of revolution.

As reflecting the current of thought that brought about this and other improvements in the French schools, we again quote from

[2] *La Mathématique,* p. 270. See also Young, J. W. A., *The Teaching of Mathematics,* p. 98.

[3] See Young, p. 98.

J. Tannery, where he speaks about general methods, in his *Notions de Mathématique* (Paris, 1903): [4]

> One has not even a slight idea of what mathematics is, one does not suspect its extraordinary scope, the nature of the problems that it proposes and solves, until one knows what a function is, how a given function is studied, how its variations are followed, how it is represented by a curve, how algebra and geometry aid each other mutually, how number and space illustrate one another, how tangents, areas, volumes are determined, how we are led to create new functions, new curves, and to study their properties. Precisely these notions and methods are needed to read technical books in which mathematics is applied. They are indispensable to whoever wishes to understand the rapid scientific movement, the manifold scientific applications of our times which day by day tend to modify more profoundly our fashion of thinking and of living.

In Germany. No such unhesitating and methodical procedure obtained in adopting the calculus as a secondary school subject in Germany and England. For one thing the central power has less authority in these countries; and some will say that the Gallic mind is more logical than the Nordic.

As far back as 1816 the Prussian *Lehrplan* for secondary schools included "analytic geometry, maxima, and the calculus." It remained a dead letter, however; for the entrance requirements to the university were much lower than the advanced courses provided for in the *Lehrplan.* The universities were hostile to the teaching of the calculus in the secondary schools and instruction in it was actually forbidden. In 1843 Schellbach, then a school teacher in Berlin, wrote a text on conics and in it he used the methods of the calculus, but avoided the conventional notation and presentation so as not to offend against the regulations of the Prussian ministry of education. In 1865 another schoolmaster, Baltzer, taught analytics and the calculus to a special class in a school in Dresden. In his *Die Elemente der Mathematik* he introduced the function idea but gave no graphic representation.

It is interesting to know that at a meeting of school teachers in Hanover in 1864 it was proposed to limit the course to constant quantities. In the discussions we learn that Bertram was at this time teaching analytics and the calculus in a Modern school (*Realschule*) in Berlin. The *Realschulen,* emphasizing mathematics and the sciences, had met with consistent opposition from university circles and state officials. After 1855, however, these schools were

[4] See Young, p. 181.

accorded more consideration by the ministry of education, and following the Franco-Prussian War they grew in importance, and with them the mathematical branches.

Advanced mathematics was not given in the *Gymnasium.* In 1873, Gallenkamp, the headmaster in a large Modern school in Berlin, pleaded with the ministry of education that analytics and the calculus be taught in the upper classes of the *Gymnasium.* But no action followed. In the *Oberrealschule* the advanced courses fared better. In the Prussian *Lehrplan* of 1880 it was permitted to teach conics analytically and the differential coefficient was at least mentioned, while analytics and the calculus were recognized as admissible, though not essential, disciplines for the *Oberrealschule.*

It was about this time that the discussion of the function concept as the centralizing notion of instruction began to be vital. In Baltzer's *Die Elemente der Mathematik* (1865) we find the function concept employed, but there is no graphic representation of functions. However, in Bardey's texts, written in 1881, the function idea and graphic representation of relations have a large place. These texts were the most popular mathematical school books of the day. But the function idea was slow in making its way. After two years' discussion by representative schoolmen, the Prussian syllabus of 1882 permitted teachers "to introduce the boys in the upper classes to the especially important idea of coördinates, and to explain to them in the simplest manner some of the fundamental properties of conics."

And so was opened a way for introducing the function idea, though no reference to it was made by any one. The function idea was to be taught in the upper classes, according to the syllabus, but it gradually worked its way to the lower classes, as is evidenced by the syllabus of 1901.

It was at this time that Felix Klein began to take a leading rôle in the reform of mathematical teaching in Germany and in the world. This noted Göttingen professor was equally distinguished as a scholar in pure mathematics and as a leader in the pedagogy of mathematics. In both of these capacities he exerted a wide influence on American mathematics through the many American college and university teachers who did graduate work at Göttingen. In 1908 he was chosen president of the International Commission on the Teaching of Mathematics.

The Prussian *Lehrplan* of 1880 *permitted* the calculus to be taught, and that only in the *Oberrealschule*. At a school conference in Berlin in 1900, Dr. Klein and others urged that elementary calculus become a regular part of the curriculum, at least in the Modern schools. The plea was disregarded.

In Germany the function concept had now become a practical question that pressed for a solution. In 1902 appeared the first mathematical school book featuring the function idea, though it was not admitted into the schools till 1906. At a meeting of the Society of Natural Scientists in Breslau in 1904 Dr. Klein claimed that "the function idea graphically represented should form the central notion of mathematical teachings, and, as a natural consequence, the elements of the calculus should be included in the curriculum of *all* 9-class schools." That is, in the *Gymnasien, Realgymnasien,* and *Oberrealschulen*. Such a course, he noted, was being given in French schools. Those present at the meeting reacted favorably to the suggestions and chose a committee to draw up reform proposals. The committee's proposals, the now historic *Meraner Lehrplan,* were adopted at the Society's meeting at Meran the following year (1905). It was the meeting at Meran that coined the expression *funktionales Denken* (functional thinking), so frequently used in recent discussions.

One result of the Meran meeting was that the Prussian ministry of education gave permission to five 9-class schools to experiment in the direction indicated. Later the plan was taken up by other schools. In answer to a circular letter of 1908 it was found that of the twenty-six 9-class schools that returned an answer (there were thirty-eight in all) thirteen had introduced the function idea and the calculus.

The Meran proposals exerted a large influence on the other German states and on Austria. The latest syllabi show that introductory calculus is now incorporated in the secondary school system throughout Germany and Austria.

In England. In England the calculus had been given in the secondary schools long before the modern movement for reforming the mathematical curriculum started. In the stronger Public Schools of England there were special classes in mathematics in the upper forms for boys ranging from seventeen to nineteen years of age and with some mathematical ability. These boys were planning either to compete in due time for the university scholar-

ships or to take the army entrance examination, or were preparing to take up engineering. The courses were much like what is ordinarily given in the sophomore year of American colleges; Todhunter was a commonly used text.

But the later movement for introducing the calculus, which rose from centering mathematical instruction around the function concept and making school mathematics apply in practical life, began in the nineteenth century. On this latter phase the well known engineer, mathematician, and educationist, John Perry, spoke and wrote. In his momentous address on the teaching of mathematics delivered at Glasgow before the British Association in 1901, he said: [5]

> As examples of methods necessary even in the most elementary study of nature I mention: the use of logarithms in computation; knowledge of and power to manipulate mathematical formulae; the use of squared paper; the methods of the calculus. Dexterity in all of these is easily learned by young boys.

The English secondary schools were considering so many reforms at that time, notably the teaching of geometry, and their relation to the Examinations, that the question of the calculus as a regular school subject did not become vital for some years. In 1908 the Fourth International Congress of Mathematicians meeting in Rome created the International Commission on the Teaching of Mathematics. The Commission was to study the mathematical teaching in different countries and report at the next meeting of the Congress, to be held in Cambridge in 1912. Felix Klein was chosen president of the Commission. The intervening four years were considerably occupied with questionnaires, reports, and discussions in teaching circles, conventions, and educational journals. In both England and America the reports of the Commission were very influential in analyzing and coördinating the methods and contents of their very heterogeneous school systems and in acquainting them with the better school organization and the more advanced pedagogic theory of France and Germany.

In this period the calculus question came up. A new angle, typically British, and one with which American educationists have a good deal of sympathy, entered into the discussion. It found its note in the expression, "Calculus for the average boy." English schoolmen were not unfamiliar with the value of such a course

[5] *Educational Review*, 1902, p. 158.

for those who wished to specialize in mathematics at the university or who wished to take up certain professions like engineering or the army, a reason that in France had weighed powerfully long before the curriculum of 1902. As we have seen, certain English schools, too, had given such courses. Nor were they inhospitable to the idea of the German reformers who saw the calculus as the fitting climax to a noble mathematical structure pedagogically planned; for the function concept, correlation, and unified courses were recognized values at this time in all countries. What appealed to them in this projected course was what may be called its cultural-practical value: the value of the calculus in integrated life. This was intimated already in the speech of Perry referred to above.

In February, 1911, C. S. Jackson read a paper to the London Branch of the Mathematical Association entitled "The Calculus as an Item in School Mathematics." The calculus that has hitherto been taught in the English schools, he said, has met with opposition from two directions: first, from the rigorists, who complain of the unguarded generalities permitted by the instructor or the text; and second, from the engineers and physicists who complain that the students are able to manipulate but do not know the underlying principles, and that the applications are too academic, dealing mostly with higher plane curves. They demanded that the calculus be taught more widely and at an earlier age and that it deal from the outset with the immediate applications to mechanics and elementary physics. Jackson contended that this can be done and that the calculus can be made a part of the normal equipment of a boy sixteen or seventeen years old. To do this he said he would bring the subject to the student in somewhat the same way as the human race has acquired a knowledge of it. He concluded with a suggestive syllabus for introducing the calculus. This begins with a problem of finite differences. After some more preliminary work, problems and slopes are taken up.

"The Democratization of Mathematical Education" was the title of the address of the presiding officer, Professor E. W. Hobson, at the meeting of the Mathematical Association on January 10, 1912. He advocated that the time saved by improvement in methods and by elimination of material hitherto considered essential "should be employed in introducing the pupils to a considerably greater range of mathematical thinking than has hitherto been

usual." He suggested "the possibility of making a rudimentary treatment of the ideas and processes of the calculus part of the normal course of mathematics in the higher classes of schools." The president suggested that the Association take up the question for study.

At the same meeting a paper on an earlier place for the calculus was read by C. V. Durell. He emphasized more than most of the English schoolmen the function concept and its culmination in the calculus, but he also enlarged on the cultural-practical values of the course and the incentive it gives the student to know, as only the calculus can tell him, what all his mathematics can do.

A paper on the work of the International Commission was read at this meeting by C. Godfrey. From this report we learn that only few schools had given introductory calculus a serious trial. The work of one such school, the Naval College at Dartmouth, was described in detail. Here the boys were about sixteen years old. Mercer's *Calculus for Beginners* was used. At the outset they used concrete illustrations such as $s = at^2$. A clear concept was paramount. Therefore they felt that in this course numerical work should not be onerous. Rate, slope, maxima and minima, and areas were recommended as suitable topics.

The paper evoked a lively exchange of opinions by those present. Many of the participants in the discussion had themselves given the calculus a trial in their respective schools.

In August of that same year, 1912, the International Congress met in Cambridge. Introductory calculus was again on the agenda and in Britain the interest in it as a general school subject was growing.

"Calculus for the average boy" meant to the English schoolmen just that. Once convinced of the value of the calculus to younger students they did not confine its instruction to those who would go to the universities or take up technological courses in higher institutions. Nor did they limit this work to those training in the conventional secondary school. As stated by Perry and running through all the subsequent agitation, it was contended that the notions and the methods of the calculus should reach all those who continued their schooling beyond elementary work.

An illustration in point is the work done by the Technical Day School of the Borough Polytechnic Institute. In a paper read before the Mathematical Association in December, 1913, Mr. W.

Knowles, of that school, tells us that the majority of his students came from the elementary schools, only a few from the secondary schools. They were boys with a mechanical or constructive bias, generally entered when they were thirteen, and spent three years in the school. Then they went to the engineering shops.

The first part of the course dealt with constructive plane and solid geometry. Then they took up the equation of a straight line, the solution of simple equations, and the use of mathematical tables, and while doing so they were given an introduction to the calculus. The third year they studied the parabola, the cubical curve, the sine curve, and the logarithmic curve; simple calculus was considered in connection with these curves, with a bias on the graphical side. The slopes of these curves were studied in three ways: (1) numerically, by finite differences; (2) graphically, by tangents; (3) in general terms, that is, algebraically. Integration was looked upon as the reverse of differentiation. Areas were found by counting squares and by Simpson's Rule.

After 1912 the new program began to take shape. In the general mathematical syllabus for pupils of the Public Schools not specializing in mathematics, drawn up in 1913 by the Public Schools Special Committee and approved by the General Committee of the Mathematical Association, the work was divided into two classes. The first class included the work that all pupils took, namely, arithmetic, geometry, algebra, and trigonometry; the second class contained courses that were alternative. The alternatives recommended were: (1) mechanics; (2) further work in algebra; (3) the calculus; (4) analytic geometry; (5) further work in trigonometry; (6) further work in pure geometry. For the very weak students the committee suggested mechanics as the easiest, and the calculus was placed next in difficulty. As to the work in these courses, it was said that "this is intended neither to be specialists' work nor to demand great mathematical ability." It was meant for boys of sixteen to eighteen years of age. They stated further: "It is desirable that such boys should work at fresh subjects involving new ideas." The corresponding committee of Other Secondary Schools (those not included under Public Schools) stated in their report: "It is desirable that boys of ordinary ability who attend a Secondary School to the age of sixteen should not leave without some introduction to the principles on which the calculus is based."

A preliminary report to the Association, published in January, 1914, and based on the replies to a questionnaire by Subcommittee A of the International Commission, formed the basis for their report at Paris in April, 1914, and stated these facts: The Examination, which at first was an obstacle to introducing the new course, had become a help in its development.—Several schools in 1914 were giving introductory calculus. The slope, velocity, small increments, maxima and minima, and simple integrals were studied. The polynomial was used principally.—The teaching was not exhaustive, nor exceedingly rigorous, but correct. "State nothing but the truth, but not necessarily all the truth" was the motto.—The integral calculus was interlaced with the differential.—The most common texts were Mercer, Gibson, and Edwards.—The movement received general support, the most powerful support coming from the engineers and scientists. As evidence it was said that "physicists have long pressed for a modicum of calculus, and prefer to take it without too much mathematical rigor." Further it was held that "whatever opposition there has been has come from those who fear that a diminished emphasis on manipulation and the formal side of algebra will have a bad effect."

In this movement there has been no backward turn. The number of schools taking up the calculus is growing. But in Britain there is no bureaucratic ministry of education to bring about a reform like this; it must run the gauntlet of popular and professional discussion. It is, however, now becoming quite general in both England and Scotland. Australia also gives the new course in certain schools. Even some Canadian schools offer the calculus, though they are governed much by the attitude of the schools in the United States.

In the United States. In the United States the teaching of the elementary calculus did not become a live question until within the last decade. And yet the subject was broached to American teachers of mathematics exactly a quarter of a century ago. In his presidential address [6] on "The Foundations of Mathematics" delivered before the American Mathematical Society in New York, December 29, 1902, Professor E. H. Moore, of the University of Chicago, said:

> As a pure mathematician I hold as a most important suggestion of the English movement the suggestion of Perry's, just cited, that by emphasizing

[6] *Bulletin of the American Mathematical Society* (1903), pp. 402-424.

steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing, and graphical methods generally, in continuous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus.

In discussing the laboratory method in the secondary school, he said:

In agreement with Perry it would seem possible that the student might be brought into vital relation with the fundamental elements of trigonometry, analytic geometry, and the calculus, on condition that the whole treatment in its origin is and in its development remains closely associated with thoroughly familiar phenomena. With the momentum of such practical education in the methods of research in the secondary school, the college students would be ready to proceed rapidly and deeply in any direction in which their personal interests might lead them.

With almost prophetic suggestiveness Moore discussed the problems of primary, secondary, and higher education, offering solutions twenty-five years ago that have become realities today,—such as correlation of subjects, unified courses, laboratory methods, junior colleges, teacher training. But his suggestion to teach in the secondary schools the advanced branches of trigonometry, analytic geometry, and the calculus met with less response—the calculus, no response. Years were to pass before any serious attempt was made to introduce analytics and the calculus into the high school curriculum. These were the years when Klein in Germany and Perry in Great Britain were awakening their co-laborers to new possibilities in the field of secondary school mathematics. But a huge democracy makes its changes slowly.

Yet there was leavening work going on, making ready even for this innovation. The many mathematical organizations that sprang up after 1902 displayed in their conventions and in their journals constructive activity. The work and reports of the International Commission had a marked effect on the teaching of mathematics in the United States. These reports, which began to appear in 1911, helped to make known the educational practices in the different parts of our country. This has helped bring about an orientation in pedagogic theory and a standardization of curriculum and organization that had hitherto been lacking. These reports also made known to American educators the work done in other countries and gave them opportunity to learn of their deficiencies—and also of their excellencies. The national Committee on Mathemati-

cal Requirements was created in 1916. This committee labored from 1916 to 1923 and performed a most monumental work.

As a result of these various agencies the graph and geometrical curves from analytic geometry have been introduced into American high school algebras—largely since 1910; and improved problem material selected from physics, mechanics, and everyday life has supplanted many of the "disciplinary" problems of the old books. The movement to make the function concept *actually* and *consciously* (and not just theoretically) the central principle of the mathematics course, with American teachers and textbooks, has been largely due to the work of the National Committee.

The Committee recommended that elementary calculus be offered as an elective in the senior high school. It made plain "that this is not intended for all schools nor for all teachers or all pupils in any school." Moreover, they pointed out that it is not to connect in any direct way with college entrance requirements. "The future college student will have ample opportunity for the calculus later. The capable boy or girl who is not to have the college work ought not on that account to be prevented from learning something of the use of this powerful tool."

The Committee justified its inclusion of this "college study" in the high school curriculum by noting the character of the study. "By the calculus we mean for the present purpose a study of *rates of change*. In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a changing quantity become largest or smallest? How can rates of change be compared?—These are some of the questions which lead us to study the elementary calculus. Without its essential principles these questions cannot be answered with definiteness."

This projected course in elementary calculus, according to the National Committee, should include:

(a) The general notion of a derivative as a limit indispensable for the accurate expression of such fundamental quantities as velocity of a moving body or slope of a curve.

(b) Applications of derivatives to easy problems in rates and in maxima and minima.

(c) Simple cases of inverse problems; for example, finding distance from velocity.

(d) Approximate methods of summation leading up to integration as a powerful method of summation.

(e) Applications to simple cases of motion, area, volume, and pressure.

Though this recommendation startled the general run of mathematics teachers who had not followed developments in European schools, the committee only gave voice to forces already at work, of which we shall mention two.

One factor that has for years been influential in bringing out the pedagogic possibilities of the calculus is the practice that some college and university departments of mathematics have of including the calculus in their freshman courses. Among the first to give such work to freshmen were Professors F. S. Woods and F. H. Bailey of the Massachusetts Institute of Technology, and Professors A. S. Gale and C. W. Watkeys of the University of Rochester. This was about 1907. Other colleges took up the idea and since 1916 such courses have been given in several institutions.

Giving the calculus to freshmen in the colleges necessitated a change in the customary mode of presentation, which had become very "scientific" and very formal. That was a pedagogic contribution that gave much promise, and herein lay the chief value of these college experiments for high school mathematics, especially when the methods of presentation became available to other teachers through textbooks written for such courses. It was a revelation to many of us as to how adaptable to younger minds the calculus could be made, and how much easier much of the work in the calculus is than some of the work in college algebra and analytics. Thus, we soon began to consider its value in the high school.

Another, and a very direct, force emanated from Teachers College, Columbia University. Secondary school calculus had for some time interested the mathematics department at Teachers College and the experimental schools connected with that institution. Professor David Eugene Smith had been the chairman of the American committee of the International Commission and had for years watched with interest the work done in this field in the schools of France and Germany. In his courses in the teaching of mathematics he discussed the merits of the calculus as a high school study and spoke for its inclusion in the curriculum as an elective. At the time when the Report was being drawn up the experimental

schools of Teachers College were already giving the new course to their students. Their work will be described in detail later.

In 1923 the Report came out in its complete form. Since then there has been a growing interest in elementary calculus as evidenced by articles in mathematical journals, discussions at conventions, publications of texts, and now and then another school trying out the suggestion. But there is evident a lack of knowledge of the historical and pedagogical background of the question that prevents it from being discussed and judged discerningly and constructively. It is one purpose of this chapter to give the reader at least this background.

V. Description of Typical Courses Given in High Schools

Two Plans. In the few schools that have taken up this work two different plans have been followed. One plan incorporates the calculus notion and methods in the work of some other course, generally algebra, in much the same way as had previously been done with the coördinate system of analytic geometry. This has long been the method followed in France and is in line with the present movement for unifying mathematical instruction. As a modified form of this plan a year of unified or at least strongly correlated work in algebra, trigonometry, and the calculus is given in some schools. This first plan seems to be followed by schools that are definitely committed to elementary calculus. The second plan is to give it as separate work. It may be for a semester, or for a few weeks of the block of the schedule given over to algebra, solid geometry, trigonometry, and elementary calculus. For giving a new subject a place and for exploring its possibilities while it is still in the experimental stage, the second plan has some advantages over the first.

Textbooks. The matter of textbooks is an important question. The instructors who have thus far given these courses are men and women who are competent to carry on the work either with a poor textbook or with no text at all. But that will not be the case with the general run of schools. If this work is to make headway in the traditional high school, there must be suitable textbooks. The conventional college text will never do. There are a few British textbooks that are quite usable, but they look unfamiliar to American eyes. Fortunately some very good texts for college freshmen have made their appearance, and are easily adapted to high school

classes. We would mention the following, in order of publication: Smith and Granville, *Elementary Analysis* (1910), Ginn & Co.; Gale and Watkeys, *Elementary Functions and Applications* (1920), Henry Holt & Co.; Griffin, *Introduction to Mathematical Analysis* (1921), Houghton Mifflin Co.; Mullins and Smith, *Freshman Mathematics* (1927), Ginn & Co.

In the following paragraphs will be described briefly the work taken up in some of the schools giving the calculus, as it has come to the knowledge of the writer through private letters and public reports.

The Horace Mann School for Girls. Pioneer work in this field was done by the Horace Mann School for Girls, the Horace Mann School for Boys, and the Lincoln School of Teachers College.

In 1921 Miss Vevia Blair taught for the first time a short course in the differential calculus to a class in the tenth grade. The work in the calculus was part of a general mathematics course designed especially for a selected group of pupils. Her aim was to construct a course which should give to pupils, girls especially, the sort of mathematics best adapted to their ages, interests, and intellectual endowments.

The group was a college preparatory class. After a year in this general mathematics, which included some analytics, a great deal of trigonometry, and a fair amount of algebra, the class finished up plane geometry and algebra and passed off their college entrance examinations at the end of the eleventh year. For three periods a week Miss Blair and her girls together worked out a course in differential and integral calculus. No text was used, but material was drawn from several texts.

This course, with much the same syllabus, has been given as a senior elective every year since then, and covers the differentiation of the functions $f(x) = u^n$, $f(x) = uv$, $f(x) = \frac{u}{v}$, $f(x) = \sin x$, $f(x) = \cos x$, with problems on rates and maxima and minima. In integral calculus they take up problems of area, volume, work, and water pressure. In differentiation the extent of the difficulty is to be found in the development of the formula

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots;$$

and in integration the limit of difficulty may be found in calculating

the volume of a ring made by the circle $x^2 + (y - 7)^2 = 9$ in revolving about the X-axis, the integral of $y = \sqrt{a^2 - x^2}$ being given.

This year (1927-28), Miss Blair is trying the experiment of teaching the same course to a class of selected juniors. The course is given as a demonstration course in Teachers College and is open to students in the mathematics department of Teachers College.

The Lincoln School of Teachers College. Pioneer work in this field was also done by the Lincoln School of Teachers College in 1921. To begin with, the experiment was conducted within the schedule allotted to mathematics in the regular course, namely, in the period given in the eleventh year to trigonometry and the third semester of algebra. Concerning this beginning Dr. Vera Sanford, who then instructed the group, reports:

> We first took up the work in 1921-22, and for three years carried on with a group of eleventh graders, teaching them to differentiate algebraic functions, especially those involved in problems dealing with maxima and minima. In the same year we studied intermediate algebra and trigonometry. Classes of about fifteen students about sixteen to seventeen years old. The entire group of that year studying mathematics—both weak and good—was in the class. The work was not elective, but was needed for college entrance, as the algebra units had to be completed for most colleges. The whole group was going to college—and got there! Text—Young and Morgan at first, then Gale and Watkeys. Neither was satisfactory for the groups. Students took, and passed, Cp. 3 of the College Entrance Examination Board.

The practicality of the new course being now demonstrated in territory borrowed, as it were, from algebra and trigonometry, it was decided to give it space and place of its own. So the work in the calculus was moved up to the twelfth year and was made a part of an elective year course. The eleventh year work was reorganized so as to include the work necessary for college entrance. But it was also organized so as to make a complete course in itself if the student should not elect any more mathematics, and to form a good propædeutic course for the twelfth year work if he should continue. The first two-thirds of the year's work is occupied largely with the linear and quadratic functions and computational trigonometry. The remainder of the year is given to a unit in solid geometry. Here many of the theorems call for the summation of small elements and the limit of a sum and thus the student is informally introduced to the notion of the calculus without using its formal technique.

The actual work in the calculus begins with the twelfth year and

covers roughly the first semester. The key idea is the extension of the function concept to that of the rate of change of a variable quantity. This leads directly to the introduction of the derivative and its applications to velocities and accelerations. The indefinite integral as the inverse of the derivative is used for finding areas, volumes, momentum, force, and work. The last semester is given over to advanced algebra and further work in trigonometry; but the technique acquired in the calculus is used whenever possible. The subject matter of the course is to be found in Griffin's *Introduction to Mathematical Analysis,* Chapters III and IV, and in Gale and Watkeys' *Elementary Functions and Applications,* Chapters VI and VII.

Mr. Gordon Mirick, who has had charge of these groups, and Dr. Vera Sanford, of the Lincoln School, have described the course in detail in *The Mathematics Teacher,* Vol. XIX, No. 4.

Wadleigh High School, New York. Mr. John Swenson of the Wadleigh High School of New York has for some years tried the experiment of combining the calculus with advanced algebra, trigonometry, and solid geometry. Differentiation and integration of algebraic polynomials are introduced in the advanced algebra given the latter half of the eleventh year. The derivative is used to find maxima and minima in problems, to obtain the equations of tangents and normals to conics, and to determine the points of inflection in curves. Areas are obtained by integration. In the twelfth year the treatment of trigonometric functions is taken up in connection with the formal study of trigonometry, and trigonometric substitution is used to integrate algebraic expressions. While taking up the formal study of solid geometry, they integrate to find volumes and surfaces. This work is described more fully in the National Committee's Revised Report.

It seems, from the Report, that advanced algebra, trigonometry, and solid geometry have their usual portion of the program, and that the calculus goes through it all, the great unifier. As one of our number says of the calculus: [7] "It unifies all mathematics, it brings together the loose ends and gives the teacher or student unified or grouped wholes, and in doing so merely repeats human experience."

The University High School, Oakland, Calif. Since the fall of 1925 the University High School of the University of California

[7] Byron Cosby, in *The Mathematics Teacher,* Vol. XVI, p. 433.

has given to twelfth year students a course in the calculus combined with solid geometry. The plan of the curriculum follows.

The tenth year's work includes all of plane geometry and five weeks' work in solid geometry—emphasis in the latter being shifted from demonstration to computation. This seems a highly desirable plan since the large majority of the pupils take no more mathematics. It affords also extra time for the mathematically capable who elect full four years of mathematics. The third year's work includes algebraic theory, trigonometry, and some work in the natural sciences. In the fall term of the senior year they take up elementary calculus and the more demonstrative parts of solid geometry. About ten weeks' time is given over to the calculus, since considerable solid geometry has been studied in the tenth year. Following the recommendation of the National Committee, the treatment of the mensuration theorems of solid geometry is simplified by applying Cavalieri's and Simpson's theorems and the methods of the calculus. Analytic geometry is offered the second semester of the senior year.

Miss Ethel Durst has selected and planned the work in the calculus, and Miss Gertrude E. Allen has done the same for analytic geometry. We quote from a letter describing the work: "We do not offer college courses but emphasize the concrete approach and the interesting and obvious applications. We have good sets of practical problems gleaned from all the sources we could find, and I think it is this phase of the work that makes a special appeal."

The course is fully described by Miss Durst in the *University High School Journal,* June, 1926. Miss Durst's excellent syllabus is included in full. From the introduction we quote, as pertinent to our topic:

> America has demanded an equality of opportunity in education, and the schools are confronted with a consequent inequality of ability. . . . In the interests of the average we have attempted a psychological approach to so logical a subject as geometry, and have eliminated most of the abstractions from so abstract a subject as algebra. But has as careful provision been made for the gifted child? In mathematics, at least, may there not be some justification for the psychologists' assertion that "the superior children are our most shamefully neglected children," that "the greatest retardation and educational waste exists among the brightest pupils"?

The University High School of the University of Iowa. A very recent school to introduce the calculus is the University High School, of Iowa City. In September, 1927, a class of eleven was

enrolled—six boys and five girls. The way in which the course was inaugurated as well as the manner in which it is conducted probably represents how such courses will be inaugurated and conducted when they shall become more general. For here there was no particular trepidation about offering this course: the pedagogic value and general feasibility of such a course seemed to those in charge pretty well established. It was a question of fitting it in with this particular school's needs and resources. Here was a select group—selected, not by the school, but by their own interests and abilities and future plans—who wanted an advanced course in mathematics. The course could have no bearing on college entrance, for they had already had three semesters of algebra; and provision had been made for solid geometry. They wished to have a course that would be of value to them if they did not take up college work, and which should help them in their work in mathematics, in science, or in engineering in case they went to college. The school had formerly given a fourth semester of algebra, and it was this time decided to try a course in elementary calculus and related material.

The man in charge of the mathematics work had much other work to do and, besides, had had no experience in teaching this particular course. So it was decided to use a basic textbook (Griffin) and to follow it quite definitely to the same extent as is done in other high school subjects. They plan to cover in five days a week for a semester a little less than what college freshmen using that text cover in four days a week. The text is so well known that there is no need of describing the contents of the course. Supplementary material from other texts is used, but the order of the text is followed.

The instructor, Mr. Frank Austin, reports that the students show keen interest and are much impressed by the power of the "new mathematics." They are especially struck by the ease with which they now can work the problems in their physics course. The writer read over the students' replies to a questionnaire on the difficulties of the course and their reactions to the future and immediate uses of the subject. From these replies and from other reports it seems to be the consensus of opinion that this is "the most useful and helpful mathematics" that they have ever had.

The New Britain (Conn.) High School. The senior high school in New Britain, Conn., began giving a course in elementary

calculus as a senior elective in 1922-1923, following closely the recommendations of the National Committee. This school uses the laboratory method in its mathematical instruction, and this plan was also used in the calculus course. Mr. Robert R. Goff, who has had charge of the work, writes: "There were no marks and no assignments. We worked informally after the laboratory method. A point, however, which I wish to emphasize is that the students were active and fairly successful, both in attacking the problem and in its solution. They were by no means overwhelmed by the abstractness of the subject."

The course is an elective for seniors who can meet certain requirements in their junior mathematics, and the work has hitherto been done outside of school hours. In the several years since 1922-1923 the enrollment has varied from 10 to 17. In New Britain the calculus has been given as an independent course, that is, it has not been given in connection with algebra or any other high school subject. The three main topics taken up in the course are variable rates of motion, maxima and minima, and areas and volumes. About twenty-five interesting problems are collected for each group. Enough theory and exercises are then given to prepare for the problems.

Mr. Goff has prepared his own text, *Twenty Lessons in Calculus*, mimeographed by the New Britain High School. For the benefit of those interested in this little text I quote the chapter headings: 1. First Principles. 2. Finding dy. 3. The Derivative. 4. Rate of Motion. 5. Maxima and Minima. 6. The Second Derivative. 7. Integrals. 8. Distance, Area, and Volume.

Other Schools Teaching the Calculus. The work in these six schools is described somewhat in detail so that the reader may see some typical varieties of content, arrangement of subject matter, and method of presentation employed in giving this new high school subject. Other schools have given the elementary calculus, but those discussed above are at least types of experiments that are being tried in America.

VI. Conclusion

A Suggestion. The writer holds that with the possible exception of numerical trigonometry there is no branch of mathematics which can be given to people of high school age that will appeal to them so strongly as elementary calculus. No other high school mathe-

matics will take them so intimately into their material environment. The calculus has been called the mathematics of nature. Nature is dynamic—not static. "In nature all things change." For studying these changes quantitatively the calculus uses a notation and a technique more economical and beautiful than those used in more elementary branches. Such quantitative changes the student will meet on every hand, and if he has the knowledge he will often wish to solve them. Take such problems as these:

1. A man having 20 ft. of wire netting wishes to make with it a chicken enclosure, rectangular in form. If he uses some of the barn wall for one side, what dimensions will give the largest enclosure?

2. A bomb was dropped from an airplane 2,000 ft. high. When did it strike the ground and with what speed?

3. Find the total pressure on the end of a cylindrical boiler of radius 4 ft., placed horizontally, and half full of water.

4. The baseball "diamond" is a square 90 ft. on each side. A ball was batted along the third-base line with a speed of 100 ft. per second. How fast was its distance from first base changing $\frac{2}{10}$ sec. after starting?

Furthermore, no other mathematics brings out so clearly the idea of functionality and its uses. I have before me a *Second Course in Algebra,* published as recently as 1926. Like most algebra texts that have appeared since 1923 it stresses the function concept and the utilization of the formula and the graph. It has some very interesting problem material, including statistical data. These latter and the algebraic polynomial show the uses of the graph in finding maxima and minima and interpolated values of functions. The zero values of polynomials and the intersections of geometric curves are also found graphically. And with that ends the primarily functional study of the text and, presumably, the student's training in functional thinking, unless he continues his mathematics in college.

What a pity to stop here! What lack of economy! Every student of higher mathematics knows that the function concept reaches its fruition in the infinitesimal calculus; that not until the student has looked into that wonderland of the mind does he experience the keenest pleasure in studying mathematics; that not until he has been introduced to the notion of limit and has seen the derivative give rates and slopes and the integral give areas and volumes and pressure, does he appreciate the full significance of function and the real purpose of the graph.

A word about the graph. How profligate we teachers—and our

textbooks—are with the student's time! The mechanical work involved in constructing a graph is considerable, and once that graph is completed we should make the greatest use of it. But what do we generally find? Most of the student's time is taken up in dull drudgery calculating for points and plotting them. Scarcely any time is devoted to *studying* the graph. The *uses* of the graph are not stressed. The zero value of polynomials, the intersection points of geometric curves, interpolated values, and maxima and minima are the applications usually represented. And, let it be said, they are valuable additions to the student's knowledge. But how much more satisfaction, how much greater mental stimulus, how much keener appreciation of the usefulness of the graph would come to him if he saw rates, areas, volumes, water pressure, functions he meets with every day of his life—unless he is sick in bed—computed from those same curves!

And this can be done with only a little extra time if the notions of the calculus are introduced. Schools that cannot give a separate course in the calculus can do some of this work with graphic differentiation and integration in connection with the work in algebra. This is the practice followed in French schools. In three weeks' time one can take up problems in rates, areas, and volumes—enough for a student to catch a new vision of the powers of mathematics, and enough to help him actually to solve problems in his trade or in his business, should he study no further. If the reader will but examine the first chapter in F. L. Griffin's *Introduction to Mathematical Analysis,* he will be struck by the variety of problems in graphic calculus comprised in the first fourteen lessons. And they are all within the reach of many high school students.

As in algebra, so in the calculus, the graph is but the introduction. The main purpose is to give clearness to the notions of function and of limit. For greater effectiveness, analytical processes must be studied. The student will learn, however, that there are types of problems both in algebra and in the calculus that can best be worked graphically.

The analytical methods should begin with the algebraic polynomial. The ability to find the rate of a falling body from the law $s = \frac{1}{2} gt^2$ and the height to which a projectile rises from the formula $h = v_0 t - \frac{1}{2} gt^2$ invariably gives a feeling of pleasure and newly-found power. So does his ability to do the Norman window problem, the ladder problem, the gutter problem, the boat-auto-

courier problem, and the problem of the most economical salmon can.

The few American schools that give the calculus confine themselves largely to the algebraic polynomial, sin x and cos x, but the schools of France and Britain quite frequently differentiate and integrate e^x, tan x, and cot x.

There are teachers who recognize the value of high school calculus, and who yet can find no place for it in their schedule. Solid geometry and higher algebra are given as electives and that is all they find time to teach. To these the writer would suggest a plan born of his own experience. It consists mainly in making a better use of the semester allotted to higher algebra. By a rearrangement of topics and by affording such motivation to algebraic technique as only the calculus can give, the greater portion of the conventional algebra text can be covered and considerable calculus besides.

A Plan. The writer has tried this rearrangement for three years with classes of college freshmen who entered with two semesters of algebra. The situation is not exactly parallel, of course. Theoretically at least, college freshmen are a more select group than high school seniors; and they are ordinarily one or two years older. Yet the case is not as different as it seems to be at first sight. For fully half of these students have avoided mathematics as much as possible in the high school and take it in college only to satisfy the requirement of one year of science or mathematics, and choose mathematics as the lesser of two evils. Those mathematically interested have generally taken three semesters of algebra in high school and are enrolled in a different course of freshman mathematics.

Three years ago, instead of giving the above group only higher algebra the entire first semester as had been previously done, we planned to spend over one-third of the semester on graphical work and the calculus. We worked out a plan which has been followed since.

Each member of the class buys two texts, Hawkes, Luby, and Touton's *Second Course in Algebra* and Griffin's *Introduction.* We spend about eight weeks on the algebra taking up the topics most essential for advanced work—factoring, fractions, linear equations, exponents, square roots, and radicals—at the ordinary pace. Then, instead of continuing there with the chapters on functions and graphs, we turn to Griffin for four weeks and consider these topics, bringing out the graphic application to rates, slopes, areas, volumes,

distance, water pressure, work, and momentum. The next two weeks are taken up with the notion of limit and the derivative of $y = kx^n$, where n is a positive integer, and applications of this formula to rates, slope, and maxima and minima. Then we again return to the algebra text, using it for the remaining three weeks. The accelerated energy in acquiring algebraic technique and the increased interest in algebraic theory are quite noticeable as the class begins the second installment of topics in algebra. For their work in the calculus has shown them what they can do with their mathematics; in the meantime they have also had occasion to realize their deficiencies in factoring, fractions, and expansion of binomials, and these operations now take on new values.

Omitting the topics in the algebra text that have been taken up in Griffin by this time, the remaining three weeks are given over to imaginaries, the binomial theorem, the theory of quadratic equations, and simultaneous quadratic equations. The only topics omitted in the conventional text are ratio and proportion, progressions, logarithms, and symmetric systems of quadratic equations. Some topics have to be omitted, of course; and we have concluded that the substitution of the work in the calculus for these latter topics is a profitable one.

We think the plan in modified form is worth trying by instructors teaching higher algebra. For the benefit of explorers in this field we give the following references with time used in covering each: Hawkes, Luby, and Touton, pp. 1-128 (8 weeks); Griffin, pp. 1-98 (6 weeks); Hawkes, Luby, and Touton, pp. 144-147, 227-234, 153-183, 188-193 (3 weeks). As was stated before, the situations are not exactly parallel. And yet, as between the mathematical interests and attainments of a class in high school electing the third semester of algebra and of a group of college freshmen of whom at least half take it as an alternative requirement, it is not the high school class that will suffer by comparison.

Summary. Functionality is exhibited by graphs and equations giving a general idea of the relation existing between variable quantities. But one has not an intimate understanding of the function until he has some means of comparing the rates of change in the quantities. To give a clear and compact method of doing this and to utilize this method and its inverse operation is the valuable addition contributed by the infinitesimal calculus to the mathematical curriculum. If the function concept is to be the central notion in all

our mathematical teaching, then the course in secondary school mathematics is not rounded off until, to the elementary subjects of algebra and geometry, we add the fundamental notions of the calculus.

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SELECTED TOPICS IN CALCULUS FOR THE HIGH SCHOOL

By JOHN A. SWENSON
Wadleigh High School, New York City

I. Recent Movements

The International Commission on the Teaching of Mathematics. At the opening of the present century, interest in mathematics was stimulated by a number of international mathematical congresses. The first of these to concern itself with the teaching of mathematics was the Fourth International Congress of Mathematicians held in Rome, in April, 1908. This congress was attended by five hundred and thirty-five members representing twenty-two different countries, and led to the creation of The International Commission on the Teaching of Mathematics. The American commissioners consisted of Professors D. E. Smith of Teachers College, Columbia University, W. F. Osgood of Harvard University, and J. W. A. Young of University of Chicago.

The International Commission published a number of reports showing the nature of the work done in mathematics in schools throughout the world. One of the most interesting of these as far as secondary teachers are concerned is the report by J. C. Brown describing the Curricula in Mathematics in the countries represented in the International Commission.

These reports were published by the United States Bureau of Education in the years 1911-1918 and their effect on the teaching body of this country is described by Professor D. E. Smith (in the report already referred to) as follows:

> They served to show to our schools the range of our system of instruction in mathematics and the general purposes in view in the various types of school. Perhaps the chief value to our country, however, was the comparison which was thus made possible between the work done here and that done in the other leading countries of the world. This showed that we were distinctly behind other countries, as to subject matter, particularly after Grade IV, although we might properly claim to be at least equal to them in the spirit of the work done in our schools. It raised the question, however, as to

whether a good spirit could compensate for poor work, and it caused a large amount of discussion in bodies of teachers throughout the country. The past ten years have shown some gratifying results of this discussion.

National Committee on Mathematical Requirements. The influences described above set the American mathematicians and teachers to work. The Mathematical Association of America was formed in 1916. It was this association which sponsored the National Committee on Mathematical Requirements and its president, Professor E. R. Hedrick, appointed the original members of this committee. The committee's report, "The Reorganization of Mathematics in Secondary Education," appeared in 1922. A revision of the report has been published by Houghton Mifflin Co.

This report was clearly influenced by the earlier reports and makes recommendations which when adopted will bring us into closer agreement with the mathematical curricula of the European countries already referred to. In the words of Professor D. E. Smith:

> It set forth very clearly the aims of mathematical instruction in the several years of the junior high school, the senior high school, and the older type of four-year high school. It presented the model courses for these several types of school and made suggestions for carrying out the work. It considered the question of college entrance requirements, the basal propositions of geometry, the rôle of the function concept, and the terms and symbols which might properly have place in the schools. It fostered various other investigations, including the present status of the theory of disciplinary values, the theory of correlation applied to school grades, a comparison of our curricula with those in use abroad, experimental courses in mathematics, standardized tests, and training of teachers.
>
> It is not too much to say that the advance in the teaching of mathematics in our secondary schools in the last decade has been due in large part to the work of this committee.

The National Committee has recommended four plans which may be suggestive and helpful to teachers in arranging their courses.

The Different Plans Recommended. In its final report, issued in 1922, the National Committee on Mathematical Requirements gives four plans for mathematical courses for the years ten, eleven, and twelve.

PLAN A

Tenth Year:	Plane demonstrative geometry, algebra.
Eleventh Year:	Statistics, trigonometry, solid geometry.
Twelfth Year:	The calculus, other elective.

Plan B

Tenth Year:	Plane demonstrative geometry, solid geometry.
Eleventh Year:	Algebra, trigonometry, statistics.
Twelfth Year:	The calculus, other elective.

Plan C

Tenth Year:	Plane demonstrative geometry, trigonometry.
Eleventh Year:	Solid geometry, algebra, statistics.
Twelfth Year:	The calculus, other elective.

Plan D

Tenth Year:	Algebra, statistics, trigonometry.
Eleventh Year:	Plane and solid geometry.
Twelfth Year:	The calculus, other elective.

The recommendation for the twelfth year is identical in each of these four plans. This certainly indicates that the committee had no doubt as to what the work for the twelfth year ought to be in the ideal course in mathematics.

The recommendations of the National Committee were made in close coöperation with bodies of teachers throughout the country and are the expressions of national rather than local opinions. Both the colleges and the secondary schools were represented on the National Committee so that its report does not represent undue dictation from either of these two institutions.

Under these circumstances, we can do no better than to quote the reasons given by the National Committee for recommending the calculus as a high school study:

In connection with the recommendations concerning the calculus, such questions as the following may arise: Why should a college subject like this be added to a high school program? How can it be expected that high school teachers will have the necessary training and attainments for teaching it? Will not the attempt to teach such a subject result in loss of thoroughness in earlier work? Will anything be gained beyond a mere smattering of the theory? Will the boy or girl ever use the information or training secured? The subsequent remarks are intended to answer such objections as these and to develop more fully the point of view of the committee in recommending the inclusion of elementary work in the calculus in the high school program.

By the calculus we mean for the present purpose a study of rates of change. In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a changing quantity become largest or smallest? How can rates of changing quantities be compared?

These are some of the questions which lead us to study the elementary

calculus. Without its essential principles these questions can not be answered with definiteness.

The following are a few of the specific replies that might be given in answer to the questions listed at the beginning of this note: The difficulties of the college calculus lie mainly outside the boundaries of the proposed work. The elements of the subject present less difficulty than many topics now offered in advanced algebra. It is not implied that in the near future many secondary school teachers will have occasion to teach the elementary calculus. It is the culminating subject in a series which only relatively strong schools will complete and only then for a selected group of students. In such schools there should always be teachers competent to teach the elementary calculus here intended. No superficial study of calculus should be regarded as justifying any substantial sacrifice of thoroughness. In the judgment of the committee the introduction of elementary calculus necessarily includes sufficient algebra and geometry to compensate for whatever diversion of time from these subjects would be implied.

The calculus of the algebraic polynominal is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, and of velocity may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations—and such training is of value to everyone.

To meet some of the objections to the teaching of the calculus by former students of mathematics we again quote Professor D. E. Smith:

If it were suggested that we should teach analytic geometry and the calculus in the high school there would be those who would recall their own work in these subjects in the sophomore year in college, and would often lament at the dullness and aridity of the teaching. They would then say that such subjects are entirely unsuited to the students in Grade 12. What they should say, however, is that the teaching they had in college is unsuited, not the essential parts of the subjects themselves.

The fact is, we already teach a certain amount of analytic geometry in every modern high school in this country, namely, when we teach the graphs of equations. Like all other subjects, they may be made too hard for the pupils or they may be made so simple as to be perfectly suited to the kind of student that will be permitted to continue mathematics in Grade 12.

Use of the Calculus in the Social Sciences. The calculus was formerly considered necessary only for the student of physical sciences and engineering but the increasing use of mathematics now made in the social sciences makes the calculus necessary for every

advanced student of science, physical or social. The concept of rate of change pervades statistics throughout, and a clear notion of it as gained through a study of the calculus is indispensable to every student of social science.

But it should be kept in mind that with the calculus as with any branch of learning, its practical and cultural value must be measured not by whether one can get along without it if one does not know it, but by whether and how much of it one will use if one does know it.

Personal Experience. As already stated, the National Committee's recommendations were made in close coöperation with various teachers' organizations throughout the country, and it was my good fortune to belong to a mathematical society at Columbia University that met every month, under the guidance of Professor David Eugene Smith, to consider the tentative reports of the National Committee. This was in 1920 and I immediately decided to try out these recommendations in the Wadleigh High School, New York City.

I began with algebra, demonstrative geometry, and numerical trigonometry of the right triangle in the ninth year. Finding this recommendation practical and decidedly worth while, I gradually kept enlarging the course so as to include more and more of the National Committee's suggestions, and since 1924 all of the topics, including the calculus, have been taught in the Wadleigh High School.

The four-year mathematics curriculum in Wadleigh High School is divided into eight terms, one term being half a school year or about nineteen weeks of instruction. In this program of eight terms, the various branches are distributed as follows:

First and Second Terms:	Elementary algebra, demonstrative geometry, numerical trigonometry according to the time allotment recommended by the National Committee.
Third and Fourth Terms:	Elementary algebra, demonstrative geometry. The time allotment is about 5:1 in favor of geometry. Some trigonometry is used in finding areas, and logarithms are used in computation.
Fifth Term:	Intermediate algebra. This term is omitted by some of the brighter pupils.
Sixth Term:	Advanced algebra.
Seventh Term:	Trigonometry.
Eighth Term:	Solid geometry.

The work in the calculus begins in the sixth term in connection with advanced algebra and in this term confines itself entirely to algebraic expressions. The calculus of the trigonometric functions is studied in the seventh term in connection with the formal study of trigonometry. Volumes and surfaces are dealt with in connection with solid geometry in the eighth term. In this way the work is spread out and made to clarify the various topics under discussion.

The presentation of the work in this fashion is made necessary by the examination requirements in New York State. The pupils have to take a state examination in plane geometry at the end of the fourth term (tenth year), one in intermediate algebra at the end of the fifth term (first half of eleventh year), one in advanced algebra at the end of the sixth term (end of eleventh year), one in trigonometry at the end of the seventh term (first half of twelfth year), one in solid geometry at the end of the eighth term (end of twelfth year).

From the examination requirements stated above, it is easily seen that this experiment in Wadleigh High School has been carried on under most adverse circumstances. But the contention of the National Committee that "calculus necessarily includes sufficient algebra and geometry to compensate for whatever diversion of time from these subjects is necessary," has been completely justified. The power gained through the algebra involved in the calculus has enabled the students to pass the old-type Regents Examination in advanced algebra.

Some of the topics taught as well as the method of approach are given in the pages that follow. But space allows only a very fragmentary treatment of this extensive subject and we shall carry the subject matter only far enough to bring out what differentiation and integration really mean. An understanding of the meaning of these two operations is the real objective to be attained in a high school course in the calculus, rather than mere ability to differentiate and integrate a few expressions.

II. Preliminary Subject Matter

Importance of Preparation. The first step in taking up a new subject is to make sure that the pupil has the necessary preparation. "Nothing is more potent to establish wrong habits and distaste for a task than the chagrin of failure which results from attempting to work beyond one's powers. Interest is not a cause

of success but a symptom of good achievement, a sign that things on the whole are going well."

The readiness of a student for the calculus does not depend on whether he is in high school or in college. It depends entirely on his previous training, whether this training has been one of understanding and meaning rather than one of mere mechanical manipulation of symbols. This preliminary training has been given through a study of such topics as: The Function Concept, Variation, and Locus.

The Function Concept and Variation. The present high school curriculum in mathematics enables a student to do two things:

1. Express and discover relationships between variables and constants.
2. Apply necessary and useful transformations to these relationships.

Having set up relationships or equations between variables, the next step is to study the variation between these variables to determine exactly how a change in one variable affects another. The calculus is the branch of mathematics which concerns itself with an exact quantitative determination of this variation between variables. Given a definite relationship or equation between two variables, the calculus enables us to determine the ratio of a change in one variable to the corresponding change in the other variable.

As an illustrative example consider the following problem: The hypotenuse of a right triangle is constant; what is the equation that connects the variations between the other two sides? If the legs of a right triangle are represented by the variables x and y and the hypotenuse by the constant c, we know from the Pythagorean Theorem that

$$x^2 + y^2 = c^2.$$

By the methods of the calculus, we obtain the equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Here dy means a change in y and dx the corresponding change in x. The minus sign shows that an increase in one variable causes a decrease in the other when the hypotenuse remains constant, an obvious geometric fact. If we have a definite triangle with $x = 5$

and $y = 12$, and x and y are allowed to vary but the hypotenuse is constant, then

$$\frac{dy}{dx} = -\frac{5}{12} \text{ or } dy = -\frac{5}{12}\,dx.$$

This last equation says that the change in y is $\frac{5}{12}$ of the corresponding change in x but in the opposite sense.

Objectives. It is evident from the example just given that the calculus concerns itself with variation. As a preliminary to the exact methods of the calculus, it is very useful and necessary to make a general study of relationships and the idea of dependence, or what is ordinarily known as the function concept. It is very important to be able to tell:

(a) On what other variables a certain variable depends.

(b) When two variables vary in the same sense or in the opposite sense.

(c) How the various parts of a simple geometric figure vary together.

Illustrative Examples. These objectives have been realized through exercises of the following type:

1. Write the formulas from geometry which will show that
   (a) the area of a square is a function of the side.
   (b) the side of a square is a function of the perimeter.
   (c) the side of a square is a function of the area.
   (d) the perimeter of a square is a function of the area.

Explain why each formula shows a functional relationship between the variables involved.

2. Complete the following statements and express the functional relationship by using symbols:
   (a) the distance covered by a man walking at the rate of 4 mi. an hour is for one thing a function of ....
   (b) the interest earned by $100 is a function of ....
   (c) the nth term of an arithmetic progression is a function of ....
   (d) the hypotenuse of a right triangle is a function of ....

3. If $f(x) = 3x^2 - 5x + 1$, find $f(2)$, $f(0)$, $f(\frac{1}{2})$, $f(-2)$, $f(a)$, $f(-x)$, $f(x+2)$, $f(\sqrt{x})$, $f(x+h)$, $f(x+h) - f(x)$.

4. Which of the following fractions are functions of x? In each case give a reason for your answer:

(a) $\dfrac{5x-10}{2x-4}$. (c) $\dfrac{ax-a+2x-2}{ax-a-x+1}$.

(b) $\dfrac{3x-4}{2x-6}$. (d) $\dfrac{5x-7y}{2x-2.8}$.

5. If in $\triangle ABC$, side AC and $\angle A$ remain constant, how does AB vary (increase or decrease) as $\angle C$ increases? How are BC and $\angle B$ affected in such a case?

6. State three theorems in geometry which involve functional relationship and state as far as you are able how a change in one magnitude affects another.

7. If in the equation $m = \frac{1}{n}$, n is doubled, how is m affected?

8. If in the formula $E = IR$, I and R are both tripled, how is E affected?

9. If x is increased by 10%, by how many per cent is (1) x^2, (2) x^3, (3) x^n increased? Answer the same question if the word "increased" in both places is changed to "decreased."

10. Given $T = \frac{ab}{c} - d(e + f) + \frac{g}{h - 1}$. If a increases and the other letters in the right member remain constant, how is T affected? Discuss the other letters in the same way.

11. If $y = 2^n$, how is the value of y affected when n is increased by (1) 2, (2) 3, (3) k? When n is decreased by 1?

12. In the right triangle ABC, which of the following are functions of angle A: the area of the triangle, $\frac{b}{c}$, $\frac{c}{b}$, $\frac{b}{a}$, a, b, c, B, C; the ratio of the segments into which the bisector of the right angle divides the hypotenuse; the ratio of the segments into which a median divides any side?

13. In the right triangle ABC, which of the following are functions of $\frac{a}{c}$: the area of the triangle, a, b, c, A, B, C, $\frac{b}{c}$, $\frac{a}{b}$, $\frac{c}{a}$, median to b, altitude to a, bisector of C; the ratio of the segments into which the bisector of B divides AC; the ratio of the segments into which the altitude to the hypotenuse divides the hypotenuse?

14. Complete the following: In any right triangle, any ratio of two sides is a function of ....

15. Show that the length of a line-segment parallel to the base of a triangle and terminated by the other two sides varies directly as its distance from the vertex.

16. The base of a triangle is 10 and the altitude is 12. Write the equation which will express the length of any line-segment y parallel to the base in terms of its distance x from the vertex. What is the value of the constant of variation?

17. The change in speed per second varies directly as the force causing it. A freight train is drawn by a locomotive that exerts a pull of 15.5 tons. The train starts from rest and in 1 second acquires a speed of 0.5 ft. per second. Write a formula connecting force in tons with change in speed per second for this particular train. Use this formula to answer the following:

- (a) What tractive force will enable the train to gain a speed of 1 foot per second in 1 second?
- (b) What gain in speed per second will result if the tractive force of the locomotive is 20 tons?

18. The area of a section of a pyramid parallel to the base varies as the square of its distance from the vertex. If the base of a pyramid is 10 sq. in. and its altitude is 8 in., write the formula that connects the area of any cross-section parallel to the base with its distance from the base.

19. The volume of a sphere varies as the cube of its diameter. If the volume of a sphere is 20 cu. in. when its diameter is k in., find the volume of a sphere whose diameter is (a) $2k$ in., (b) $\frac{k}{2}$ in.

20. The weight of a body varies inversely as the square of its distance from the center of the earth. If a man weighs 200 lb. at the surface of the earth (4,000 mi. from the center), what would he weigh 100 mi. above the surface?

The Locus Concept. Many of the problems of the calculus are stated in geometric terms requiring a knowledge of the term *locus.* The pupil must have a clear understanding of the equation of a locus as that equation which is satisfied by the coördinates of every point on the locus, and not satisfied by the coördinates of any point outside the locus.

Thus in Fig. 1 line CD is the locus of all points the sum of whose coördinates is 10 because

(a) the coördinates of all points on CD satisfy the linear equation $x + y = 10$.

(b) the coördinates of all points not on line CD fail to satisfy the linear equation $x + y = 10$.

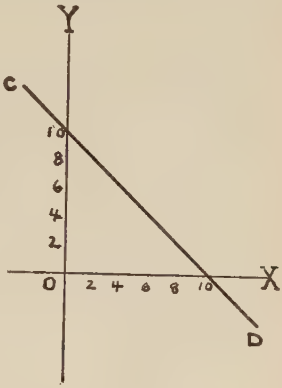

Fig. 1

A locus is a figure which contains all the points, and only those points, that fulfill a given requirement.

Illustrative Examples. A proper understanding of the term locus has been secured by exercises of the following type:

1. (a) How do we determine whether a point with given coördinates lies on a line whose equation is given?

(b) Determine which of the following points: (0,2), (2,0), $(-3,2)$, $(-3,4)$ lie on the line $2x + 3y = 6$.

2. (a) Draw a line joining the points (5,0) and (0,5). What is the sum of the coördinates of each point on this line? What is the equation of this line? Show that this equation also holds for parts of the line where one of the coördinates is negative.

(b) Proceed as in (a) but use the following points: (4,0), (0,4); (2,0), (0,2).

3. The following coördinates belong to points on CD in Fig. 1: (6,?), (4½,?) (?,7), (1½,?), (0,?), (?,0), $(-1,?)$, (5.3,?), (?,4.7).

(a) Find each point in the figure and supply the missing coördinate.

(b) What equation do all the different pairs of coördinates seem to satisfy?

(c) Extend line CD in both directions. Point out that part of CD which is the locus of points (1) both of whose coördinates are positive, (2) whose first coördinate is negative, (3) whose second coördinate is negative. What is the equation of the entire line CD? Explain why.

4. The following points lie on the line whose equation is $2x + 3y = 12$: $(2,a)$, (a,a), $(2a,a)$, $(a+3,a)$, $(3a,2a)$, $(5-a,a)$. Determine the value of a in each case.

5. Draw the line which is the locus of all points the sum of whose coördinates is (a) 10, (b) 7, (c) -4, (d) -2, (e) 0. Write the equation of each.

6. Draw the line which is the locus of all points having equal coördinates. Write the equation of the line.

7. A point moves so that its distance from the X-axis is always three times its distance from the Y-axis. Construct its path and write the equation of the path.

8. A point moves so that its distances from the X-axis and the Y-axis are in the ratio (a) $1:2$, (b) $2:1$, (c) $2:3$, (d) $3:2$, (e) $m:n$ (where m and n are two given line segments). Construct the locus in each case and write its equation.

9. If the point whose coördinates are $(x+h, y+k)$ lie on the line whose equation is $2x - 7y = 3$, show that $2h - 7k = 0$.

10. If the line whose equation is $ax + by = c$ passes through the point $(-2, -4)$, what equation must a, b, and c satisfy?

11. If the line whose equation is $y = mx + c$ passes through the two points $(2, -2)$ and $(-1, 4)$, what two equations must m and c satisfy? Find m and c and so determine the equation which passes through the two points whose coördinates are given.

12. By proceeding as in Ex. 11 find the equation of the straight line which passes through the points

(a) $(1,2)$ and $(3,4)$.
(b) $(2,5)$ and $(0,0)$.
(c) $(-2,3)$ and $(2,-3)$.
(d) $(-5,-2)$ and $(5,2)$.

13. The coördinates of a point are $(3,5)$. Find the angle that the line joining the point to the origin makes with the X-axis.

14. Draw the line which is the locus of all points

(a) whose x coördinates equal 2.
(b) whose y coördinates equal -3.
(c) whose x coördinates equal -5.
(d) whose x coördinates equal 0.
(e) whose y coördinates equal 0.
(f) 5 units distant from OX in Fig. 1.

Write the equation of each locus.

15. Draw the graphs of the following equations:

(a) $x = 1.5$.
(b) $2y + 5 = 0$.
(c) $3x = 2$.
(d) $-2y = 7$.
(e) $x = 0$.
(f) $y = 0$.

Describe each graph as a locus of points.

III. The Operations of the Calculus

Important Terms and Symbols. The mathematics preceding the calculus involves three inverse pairs of operations: addition and subtraction, multiplication and division, involution and evolution. To these six the calculus adds two: differentiation and integration. If a function is to be differentiated we indicate this by writing the letter d in front of the function. Thus $\frac{d}{dx}(x^2 + 5x + 7)$ means that the function $x^2 + 5x + 7$ is to be differentiated with respect to x. The inverse of differentiation is integration, and this operation is indicated by writing the symbol $\int$ in front of the expression to be integrated. Thus $\int (x^2 + 4x - 3)\,dx$ means that the expression $x^2 + 4x - 3$ is to be integrated.

But before the pupil can understand the two operators d and $\int$ he must learn the use of two others Δ (delta) a Greek letter which corresponds to our d and Σ (sigma) another Greek letter which corresponds to our integration sign $\int$. The operator Δ is an important forerunner of d and Σ of $\int$. Hence the importance of Δ and Σ.

The Operator Δ Used to Express Rate of Change. To draw the graph of the equation $y = \frac{2}{3}x + 1$, we may assign values to x and compute carefully the corresponding values of y. Fractional values may be avoided by giving to x values which are multiples of 3. We thus obtain the table of values given to the right. This table shows that every time y increases by an amount equal to 2, x increases correspondingly by 3, and that

x	y
0	1
3	3
6	5
9	7
12	9
:	:

$$\frac{\text{the } y \text{ change}}{\text{the corresponding } x \text{ change}} = \frac{2}{3}$$

or that

$$\frac{\text{the difference between the } y\text{'s}}{\text{the corresponding difference of the } x\text{'s}} = \frac{2}{3}$$

This ratio is known as the *change ratio* or the *difference ratio.*

The symbol Δ which we have explained in a preceding paragraph is used in mathematics as abbreviation for "difference of" or "change in." Thus the symbol Δy means the difference of the y's or the change in y, and Δx has the same meaning with respect to x. Hence the symbol for the change ratio or the difference ratio is $\frac{\Delta y}{\Delta x}$

and in the equation $y = \frac{2}{3}x + 1$ which we have considered above,

$$\frac{\Delta y}{\Delta x} = \frac{2}{3}.$$

It should be carefully noted that Δy *and* Δx are single symbols; that is, the Δ must not be separated from the x or the y. But the use of Δ is not limited merely to x and y; it may be put in front of any letter. Thus to indicate the difference of two heights we may write Δh, to indicate the difference in time we may write Δt, and to indicate difference in distance we write Δd.

Illustrative Examples. Exercises of the following type have been used to secure a proper understanding of the operator Δ:

1. An automobile driver found that at 9:10 he was 50 mi. from home and that at 9:50 this distance had increased to 80 mi. Using s (the initial letter of the word space) to represent distance and t to represent time, write the change ratio with proper symbols. What is another name for this ratio?

Solution. The given data are best arranged in tabular form as in the table at the right. From the table

Δs	s	t	Δt
30	50	9 : 10	40
	80	9 : 50	

$$\frac{\Delta s}{\Delta t} = \frac{30}{40} = \frac{3}{4} \text{ mi. per minute, or}$$

$$\frac{\Delta s}{\Delta t} = \frac{30}{\frac{2}{3}} = 45 \text{ mi. per hour.}$$

Evidently $\frac{\Delta s}{\Delta t}$ is another name for velocity.

As long as $\frac{\Delta s}{\Delta t}$ remains constant, we may say that the motion is uniform.

2. A body fell from a balloon and its velocity at various time intervals is given by the table to the right. To this table add two other columns headed Δv and Δt and fill in the values of these symbols. What is the value of the ratio $\frac{\Delta v}{\Delta t}$? If the change in velocity that takes place in one second is called acceleration and denoted by a, write a formula connecting a, v, t.

v	t
0	0
32	1
64	2
96	3
.	.
.	.

3. A body was thrown vertically upward with a velocity of 160 ft. per second and its velocity at later time intervals is given in the table to the right. What is the value of the acceleration? Continue the table four steps further, keeping in mind the body considered in Ex. 2.

Hint: Note that v is negative.

v	t
160	0
128	1
96	2
64	3
.	.
.	.

4. A body rolls down an incline with an acceleration 20 ft. per second per second. Write a table with headings Δv, v, t, Δt.

5. A and B set out from the same place, one traveling east at the rate of 20 mi. per hour and the other west at

the rate of 16 mi. per hour. At what rate are they separating? Give your answer in miles per hour. In how many hours will they be 100 mi. apart?

6. (a) Water flows into a tank at the rate of 10 cu. ft. per minute. Using V to represent volume in cubic feet, and t to represent time in minutes, write the equation connecting V and t. Also write a table with headings ΔV, V, t, Δt, assuming that the tank contains 25 cu. ft. of water at the start. What is the value of $\frac{\Delta V}{\Delta t}$, and what is the meaning of this ratio? What is the value of $\frac{\Delta V}{\Delta t}$ if t is expressed in (1) seconds, (2) hours?

(b) If the cross-section of the tank in (a) has dimensions 8 ft. by 20 ft., write a table with headings Δh, h, t, Δt, where h represents the depth of the water in the tank. What is the value of $\frac{\Delta h}{\Delta t}$? How fast is the surface of the water rising?

Write the equation connecting (1) h and t and (2) V and h. What is the value of $\frac{\Delta V}{\Delta h}$ and what is the meaning of this ratio?

Use of the Change Ratio in Graphing. To draw the graph of the equation $y = \frac{2}{3}x + 1$, let us use the table of coördinates computed on page 113. This gives us the graph shown in Fig. 2. The same result may be obtained by using the change ratio in the following manner: Starting at A go 3 units (the change in x) to the right. This gives us B. From B go 2 units (the change in y) upward. This gives us C, which is a point on the curve. By repeating the same procedure at C, we obtain as many points on the graph as we like. Instead of using A as the starting point, we may use any other convenient point on the curve. But the coördinates of A are the ones most easily obtained, because the x-coördinate of A is 0, and, for this reason, the y-coördinate of A is always equal to the constant term when the equation is solved for y. Hence the constant term shows the distance above or below from the origin to the point where the line cuts the Y-axis.

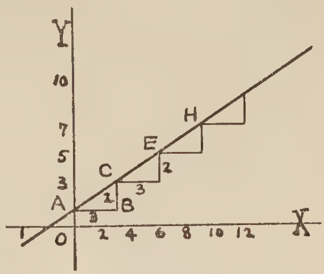

FIG. 2

EXAMPLE. To draw the graph of $y = \frac{2}{3}x - 4$:

(1) Starting at the origin go 4 units down on the Y-axis; then
(2) 3 units to the right and up 2 units.

Theoretically, it is not necessary to repeat (2), but practically it adds to accuracy and serves as a check.

EXAMPLE. To draw the graph of $y = -2x - 3$ or $y = -\frac{2}{1}x - 3$, we proceed as follows:

(1) Starting at the origin go 3 units down on the Y-axis; then
(2) 1 unit to the right and 2 units down.

Instead of (2) we may also go 1 unit to the left and up 2 units, because $-2 = \frac{-2}{1}$ or $\frac{2}{-1}$.

Illustrative Examples. The objectives of this section may be realized by examples of the following type:

1. Write down a number of equations of the form $2x + 3y = 6$ and show how to graph them by the method explained above.

2. A moving point starts from the point (3,4) and moves in such a way that y increases twice as rapidly as x. Draw the locus and write the equation.

3. Graph the equation $2x - 5y = 8$.

SOLUTION. Solving for y we have

$$y = \frac{2}{5}x - \frac{8}{5}.$$

The y-intercept is a fraction, and instead of using this point as the starting point, we may use any other point on the line which has integral coördinates, because this point is easier to locate accurately than one that has fractional coördinates. The point (4,0) is such a point; so begin here, go 5 to the right, and up 2.

4. Graph the following equations by first finding a point that has integral coördinates:

(a) $3x - 5y = 2$. (c) $7x + 2y = 4$.
(b) $4x - y = 3$. (d) $5x + 2y = 3$.

Slope of a Line. In Fig. 3 let the X-axis be horizontal and AC be parallel to the X-axis. Then AB slopes or turns away from the horizontal. The amount of this turning or sloping is measured by the angle CAB or by the ratio $\frac{BC}{AC}$. For this reason, the ratio $\frac{BC}{AC}$ is called the slope of the line AB. The angle CAB (or its equal angle XDB) is the angle that line AB makes with the X-axis. In speaking of the angle that a line makes with the X-axis we always

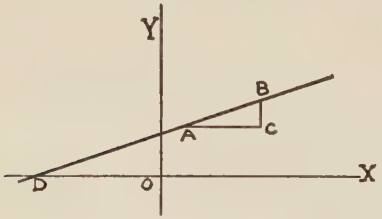

FIG. 3

mean the angle that the line makes with the positive direction of the X-axis, and not the angle it makes with the negative direction of the X-axis. Thus in Fig. 4 AB makes the angle XAB with the X-axis and not the angle CAB. The slope of line AB in this case is the ratio $\frac{BC}{AC}$. The slope is negative when the angle made with the X-axis is obtuse, because then the horizontal segment (AC in Fig. 4) extends in a negative direction. This fact is important and should be carefully remembered.

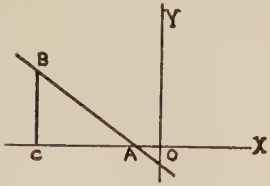

FIG. 4

It is evident that the slope of a line is always equal to the change ratio $\frac{\Delta y}{\Delta x}$. Hence the slope of a line is obtained by writing the equation in the form $y = mx + c$.

If α represents the angle that the line makes with the X-axis, then $\tan \alpha =$ the slope or $\frac{\Delta y}{\Delta x}$.

By using a table of trigonometric functions we can find the angle when the slope or the change ratio is given, and conversely.

EXAMPLE. Find the angle that the line $5x + 6y = 4$ makes with the X-axis.

Solution. Solving for y,

$$y = -\frac{5}{6}x + \frac{2}{3}$$

$$\tan \alpha = -\frac{5}{6}$$

$$= -0.8333$$

Therefore, $$\alpha = 180° - 40°$$

$$= 140°.$$

The Distance Formula. If in going from one point to another, Δy represents the change in y, and Δx represents the change in x, then

(a) $\frac{\Delta y}{\Delta x}$ is the slope of the line joining the two points and

(b) the distance d between the two points is given by the formula $d = \sqrt{(\Delta y)^2 + (\Delta x)^2}$.

The examples needed to give application of these formulas are rather obvious.

Increasing and Decreasing Functions. If y and x vary in the same sense, either variable is said to be an increasing function of the other. The condition for this is a positive $\frac{\Delta y}{\Delta x}$.

If y and x vary in the opposite sense, either variable is said to be a decreasing function of the other. The condition for this is a negative $\frac{\Delta y}{\Delta x}$.

Illustrative Examples. The understanding of increasing and decreasing functions may be secured by exercises of the following type:

1. Graph each of the following equations. Find the slope and from the slope determine whether y is an increasing or a decreasing function of x. Verify your answer graphically:

(a) $y = 0.3x + 2$.
(b) $2x + 3y = 6$.
(c) $2x - 3y = -4$.
(d) $x + y = 7$.
(e) $5y - 6x + 2 = 0$.
(f) $x + 4y = 0$.

2. Prove that the three points (2,3), (4,7), $(-1,-3)$ are collinear by showing that the slope of the segment joining the first two points equals the slope of the segment joining the last two.

3. If the three points (1,2), (3,5), (x,y) are collinear, what equation must x and y satisfy?

4. Find the equation of the straight line through the points (2,5), $(-4,3)$.

5. Show that the equation of the straight line through (x_1,y_1) having the slope $\frac{\Delta y}{\Delta x}$ is $y - y_1 = \frac{\Delta y}{\Delta x}(x - x_1)$.

6. Write the equation of the locus of a point passing through $(-3,5)$ and moving in such a way that y increases at the same rate that x decreases.

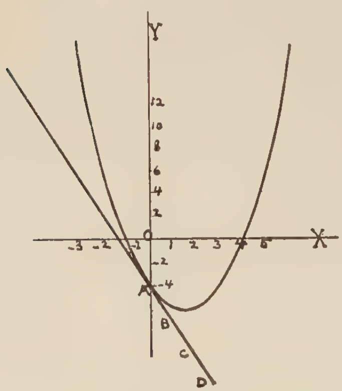

FIG. 5

The Quadratic Function. Let us consider the quadratic function $y = x^2 - 3x - 4$. If we omit the second degree term, x^2, we have $y = -3x - 4$, the graph of which is AD in Fig. 5. To obtain the graph of $y = x^2 - 3x - 4$ from the graph of $y = -3x - 4$ every y must be increased by x^2. At A, $x = 0$ and $x^2 = 0$ and A will also be a point on $y = x^2 - 3x - 4$. At B, $x = 1$ and $x^2 = 1$, so we go up one from B to obtain the corresponding point on

the parabola. At C, $x = 2$ and $x^2 = 4$, so we go up four unit spaces from C to obtain the corresponding point on the parabola. By continuing in this manner we may find as many points on the parabola as we like. It is evident that the curve will lie entirely above the line AD and be tangent to AD at A.

In the same manner we see that the graph of $y = -x^2 - 3x - 4$ will lie entirely below the straight line $y = -3x - 4$. This way of graphing is interesting because it brings out clearly how the graph of the quadratic function is related to that of the linear function. It also brings out the effect which the sign of the coefficient of x^2 has on the shape of the curve.

Let us now compute a table of coördinates for $y = x^2 - 3x - 4$, using the ordinary method of substitution, and see how x and y vary together. It is evident that $\dfrac{\Delta y}{\Delta x}$ is no longer constant but varies for different values of x. The values of x are so chosen that they form an arithmetic progression whose difference is 1. The corresponding values of y do not form an arithmetic progression but decrease for a while and then begin to increase. It is interesting, however, to note that the values of Δy form an arithmetic progression. That this fact is generally true is easily seen by using the general quadratic function $y = ax^2 + bx + c$. This fact is important in that it helps us to recognize by means of a set of values when y and x are connected by a relation of the form $y = ax^2 + bx + c$.

The following example will illustrate the main objective of this paragraph. Others of a similar type are readily made up.

Example. If $S = 1 + 2 + 3 + 4 + \ldots + n$, show by means of differences that S is a function of the second degree in n. Determine the function completely.

Solution. The table of values to the right shows that if the n's form an arithmetic progression, the ΔS's also form an arithmetic progression. Hence we conclude that $S = an^2 + bn + c$.

n	S	ΔS
0	0	
		1
1	1	
		2
2	3	
		3
3	6	
		4
4	10	

Substituting corresponding values of n and S, we obtain equations from which a, b, and c may be determined. Such equations are illustrated by the following:

$$\begin{aligned} 0 &= 0 + 0 + c \\ 1 &= a + b + c \\ 3 &= 4a + 2b + c \end{aligned}$$

Solving, we have $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 0$ and the required relation is

$$S_n = \frac{1}{2} n^2 + \frac{1}{2} n.$$

The same result may of course be obtained by using the formula for the sum of an arithmetic progression, but this method has the advantage that it applies equally well to functions of higher degrees.

The Cubic Function. To obtain the graph of the cubic function $y = x^3 - 3x - 4$, we may proceed in the same manner as we did with the quadratic function and at first omit x^3 and draw the graph of $y = -3x - 4$. We readily see that for positive values of x the graph will lie entirely above the line $y = -3x - 4$ and for negative values of x it will lie entirely below the line $y = -3x - 4$.

If we compute a table of values for this cubic function, we obtain the table given at the right. If the x's form an arithmetic progression, it can be seen that the $\Delta\Delta y$'s, or the second differences of the y's, will also form an arithmetic progression. The converse is also true, so that if the x's form one arithmetic progression and the $\Delta\Delta y$'s another arithmetic progression, the relation between y and x must be a function of the third degree in x.

x	y	Δy	$\Delta\Delta y$
-3	-22	16	
-2	-6	4	-12
-1	-2	-2	-6
0	-4	-2	0
1	-6	4	6
2	-2	16	12
3	14		

EXAMPLE. If $S = 1^2 + 2^2 + 3^2 + \ldots + n^2$, show that

$$S_n = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n.$$

Outline of the Solution. Construct a table with headings n, S, ΔS, $\Delta\Delta S$ similar to the one shown on page 119. This table will show that the relation between S and n must be of the form $S_n = an^3 + bn^2 + cn + d$. By using corresponding values of n and S, we obtain equations

$$\begin{aligned} 0 &= 0 + 0 + 0 + d \\ 1 &= a + b + c + d \\ 5 &= 8a + 4b + 2c + d \\ 14 &= 27a + 9b + 3c + d \end{aligned}$$

from which we determine a, b, c, and d.

Areas by Summation. If we graph the function $y = x^2$, we have the curve shown in Fig. 6. To find an approximation to the area

bounded by this curve and the lines OK and PK, we proceed as follows: Divide OK into n equal parts, and represent each part by Δx. Then $\Delta x = \frac{OK}{n}$. Also represent the ordinates BC by y_1, EF by y_2, . . . KP by y_n. Then the sum of the areas of the n rectangles $OBCD$, $BEFH$, and so on, constructed on the n parts into which OK has been divided, will be

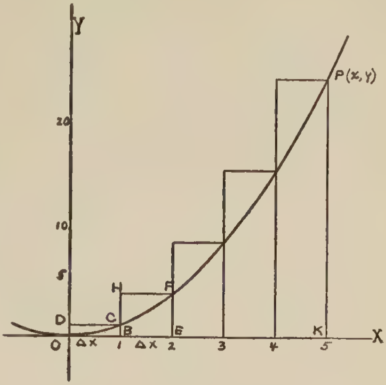

FIG. 6

$$A_r = y_1\Delta x + y_2\Delta x + \ldots + y_n\Delta x,$$

where A stands for area, and the subscript r will help us keep in mind that we have summed a series of rectangles.

Since the equation of the curve is $y = x^2$, every ordinate is the square of the corresponding abscissa. Hence

$$y_1 = (\Delta x)^2,\ y_2 = (2\Delta x)^2,\ \ldots\ y_n = (n\Delta x)^2 \text{ and}$$
$$A_r = (\Delta x)^3\,(1^2 + 2^2 + \ldots + n^2)$$

but

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

and

$$\Delta x = \frac{OK}{n}.$$

Hence by substituting we have

$$A_r = OK^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

It is evident that as n grows very large or as $n \to \infty$ (read "n tends to infinity"):

1. The value of the terms $\frac{1}{2n} + \frac{1}{6n^2}$ will approach zero.

2. The sum of the areas of the rectangles will approach the area under the curve because such areas as ODC, CHF, and so on, will approach zero. Hence we have

$$A_c = \frac{1}{3}OK^3.$$

If the coördinates of P are (x_1, y_1) the result may be written

$A_c = \frac{1}{3}x_1^3$. The subscript c will help us remember that this expression represents the area under the curve.

Volumes by Summation. Suppose we are given a cone whose base is a circle with radius 5 and whose altitude is 10. To find the volume of this cone, we begin by cutting it up into a number of thin slices each of thickness Δx. This may be done by dividing the altitude of the cone into n equal parts and passing planes through the points of division parallel to the base. Then the first section comes at the distance Δx, the second at the distance $2\Delta x$ from the vertex, and the last at the distance $n\Delta x$ from the vertex. The area of each cross-section is readily obtained by using the fact that the area of any cross-section of a cone parallel to the base varies directly as the square of its distance from the vertex. If A represents the area of any cross-section parallel to the base and x its distance from the vertex, then

$$A = \frac{\pi}{4}x^2.$$

For the particular cross-sections, we have

$$A_1 = \frac{\pi}{4}(\Delta x)^2$$

$$A_2 = \frac{\pi}{4}(2\Delta x)^2$$

$$A_n = \frac{\pi}{4}(n\Delta x)^2$$

On each section imagine a cylinder of height Δx extending upward to the next section. Then the sum of the volumes of these cylinders will be

$$\begin{aligned} S_c &= A_1\Delta x + A_2\Delta x + A_3\Delta x + \ . \ . \ . + A_n\Delta x \\ &= \frac{\pi}{4}(\Delta x)^3 (1^2 + 2^2 + 3^2 + \ . \ . \ . + n^2) \\ &= \frac{\pi}{4}\left(\frac{10}{n}\right)^3\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right) \\ &= 250\,\pi\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right). \end{aligned}$$

When n approaches infinity, the last two terms approach zero, and the sum of the volumes of the cylinders approaches the volume of the cone. Hence we have as the volume of the cone

$$V = \frac{250\,\pi}{3}.$$

The general formula is readily obtained by using in the same manner r for the radius and h for the altitude of the cone.

What $\frac{\Delta y}{\Delta x}$ Represents in a Quadratic Function. Let Fig. 7 be the graph obtained from an equation of the type $y = ax^2 + bx + c$. In passing from the point $A(x, y)$ to another point B on the same curve, let Δx represent the change in x and Δy the change in y. Then the coördinates of B will be $(x + \Delta x, y + \Delta y)$. Since B is a point on the curve its coördinates $(x + \Delta x, y + \Delta y)$ must satisfy the equation $y = ax^2 + bx + c$. Hence

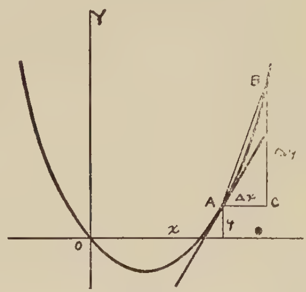

FIG. 7

$$y + \Delta y = a(x + \Delta x)^2 + b(x + \Delta x) + c$$
$$y + \Delta y = ax^2 + 2ax\Delta x + a(\Delta x)^2 + bx + b\Delta x + c$$
$$y = ax^2 + bx + c$$
$$\Delta y = 2ax\Delta x + a(\Delta x)^2 + b\Delta x$$
$$\frac{\Delta y}{\Delta x} = 2ax + a\Delta x + b.$$

$\frac{\Delta y}{\Delta x}$ represents the slope of the secant AB in Fig. 7 and the last equation is a formula that will enable us to find the slope of a secant through two given points on the curve. Any number of secants may be drawn through A and it is evident that as the value of Δx changes the slope of the secant changes. This change is due to the term $a\Delta x$. If, for example, $\Delta x = 3$, then

$$\frac{\Delta y}{\Delta x} = 2ax + 3a + b.$$

If Δx approaches 0, $\frac{\Delta y}{\Delta x}$ approaches $2ax + b$.

This last expression is the slope of the tangent at A because when $\Delta x \rightarrow 0$, the secant approaches the position of the tangent at A. To represent the slope of the tangent we use $\frac{dy}{dx}$ instead of $\frac{\Delta y}{\Delta x}$.

Hence $\frac{dy}{dx}$ represents a special value of $\frac{\Delta y}{\Delta x}$, namely, that value

which $\frac{\Delta y}{\Delta x}$ approaches when Δx and Δy approach zero. In dealing with functions of the first degree, $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$ have the same value because $\frac{\Delta y}{\Delta x}$ is constant and depends in no way on the size of Δx.

By applying the procedure given above to a few numerical examples of the type $y = 2x^2 - 4x + 1$, the pupil soon learns the distinction between $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$ and becomes ready for the mechanical method of differentiation given in the next section.

The Mechanical Way of Finding $\frac{dy}{dx}$. Thus far we have found $\frac{dy}{dx}$ by letting $x \to 0$ in the expression for $\frac{\Delta y}{\Delta x}$. We shall now consider a simple mechanical way of getting $\frac{dy}{dx}$ directly from the given polynomial without first getting $\frac{\Delta y}{\Delta x}$.

From $y = ax^2 + bx + c$ we obtain $\frac{dy}{dx} = 2ax + b$ by multiplying the exponent of x in each term by the coefficient to form a new coefficient and diminishing the exponent by 1, retaining the base. Thus $2ax$ may be obtained from ax^2 by

(1) multiplying the coefficient a by the exponent 2, and
(2) retaining the base x, and
(3) lowering the exponent 2 by 1.

From bx^1 we obtain in a similar manner $1bx^0$ or b, and the term c or cx^0 becomes 0 because c times 0 is 0. Hence any constant term in the polynomial always becomes 0 in $\frac{dy}{dx}$.

This process is general and applies to any polynomial, whether the exponents are integral or fractional, positive or negative. The process is known as differentiation, and the new function obtained is called the derivative of the given function.

$$\text{If} \quad y = 5x^{\frac{2}{3}} - \sqrt{x} + 3$$
$$\frac{dy}{dx} = \frac{10}{3}x^{-\frac{1}{3}} - \frac{1}{2}x^{-\frac{1}{2}}.$$

EXERCISES. It is a comparatively simple matter to write down a few polynomials with the different kinds of exponents and have the pupil practice differentiation until he has necessary skill.

The Meaning of $\frac{dy}{dx}$. We saw on page 124 that in connection with the curve $y = ax^2 + bx + c$, $\frac{dy}{dx} = 2ax + b$ is the general expression that represents the slope of all tangents that can be drawn to the curve. To find the slope of the tangent to the curve at a given point of tangency, we merely substitute the value x has at that point in the expression for $\frac{dy}{dx}$.

EXAMPLE. Graph the parabola $y = x^2 - 2x + 4$ for integral values of x from -3 to $+5$ and find the slope of the tangent to the curve at each of these points.

Solution. By differentiating the function $y = x^2 - 2x + 4$, we get

$$\frac{dy}{dx} = 2x - 2.$$

x	y	$\frac{dy}{dx}$
-3	19	-8
-2	12	-6
-1	7	-4
0	4	-2
.	.	.

By substituting the values of x in the given equation, we have the corresponding values of y. By substituting the value of x in the expression for $\frac{dy}{dx}$ we have the slope of the tangent at the point in question. By drawing through all such points lines which have had their slopes computed, we actually find that they are tangent to the curve. This verifies our slopes as given in the third column. We may then ask this question:

At what point will the tangent to the curve have the slope 1? This question is answered by solving the equation

$$2x - 2 = 1$$
$$x = \frac{3}{2}.$$

Illustrative Examples. Exercises of the type given below are readily made up and solved until the pupil has the necessary skill. The method applies equally well to functions of higher degrees.

1. In the equation $y = 20x + 2x^2$ find the value of x when (a) y is increasing 100 times as rapidly as x, (b) y and x are changing at the same rate, (c) x increases 50% more rapidly than y.

Hint: In (a) $\frac{dy}{dx} = 100$.

2. Determine how y and x vary together in $y = x^2 - 5x + 6$. That is, determine for what values of x the two variables vary (a) in the opposite sense, (b) in the same sense. Check by graphing.

Solution. $\frac{dy}{dx} = 2x - 5$. From the work on page 118 we readily infer that y is an increasing function of x, or y varies in the same sense as x, when $\frac{dy}{dx}$ is positive. $2x - 5$ is positive when x is greater than 2.5. When x is less than 2.5, $\frac{dy}{dx}$ is negative and y is a decreasing function of x. What happens when $x = 2.5$ we shall consider in the next section.

Maximum and Minimum. One of the most important uses of the ratio $\frac{dy}{dx}$ is to find the lowest point of the curve when the curve has a lowest point, and to find the highest point of the curve when the curve has such a point. As Fig. 8 shows, the tangent to the curve at these points is parallel to the X-axis and therefore the value of $\frac{dy}{dx}$ at either the highest or the lowest point is 0.

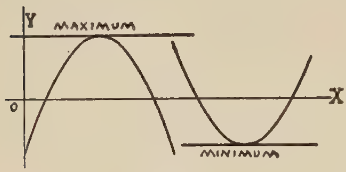

Fig. 8

At the lowest point of the curve, the function (or y) has its smallest or minimum value; at the highest point of the curve y has its greatest or maximum value. Hence the x which yields either the maximum or the minimum value of the function is found by solving the equation $\frac{dy}{dx} = 0$. It is also evident from page 119 that a quadratic function has a lowest point or a minimum value if the function has a positive second degree term; and that a quadratic function has a highest point or a maximum value if the function has a negative second degree term.

Thus we know that $y = 2x^2 - 5x + 3$ will yield a lowest point and a minimum value of the function, while $y = -2x^2 - 5x + 3$ has a highest point and maximum value of the function.

Hence in dealing with functions of the second degree, we know exactly when to expect a maximum or a minimum. But in dealing with functions of higher degrees the situation is not so simple and we need other methods to distinguish between a maximum and a minimum. As preliminary to more advanced work in this line, the pupil should satisfy himself by a careful study of graphs that the

following things are true in connection with every parabola which he encounters in this kind of work:

A. At any point on the parabola situated to the left of a maximum point:

(1) The tangent to the curve makes an acute angle with the X-axis and for this reason,

(2) $\frac{dy}{dx}$ is positive for all such values of x and

(3) y is an increasing function of x.

B. At any point on the parabola situated to the right of a maximum point:

(1) The tangent to the curve makes an obtuse angle with the X-axis and for this reason,

(2) $\frac{dy}{dx}$ is negative for all such values of x and

(3) y is a decreasing function of x.

For a minimum point the conditions are exactly the reverse; that is, positive must be changed to negative, "increasing" to "decreasing," and acute to obtuse, and conversely.

To make this perfectly plain to himself, the student should write out these conditions in full as has been done above, and save them for future reference.

Illustrative Examples. The following examples are typical of the kind that may be assigned at this stage:

1. Find the greatest rectangular area that can be enclosed by a fence 100 ft. in length. Also point out from the nature of the function involved why we are to expect a maximum.

2. Find the minimum value of the function $2x^2 + 4x - 1$. Explain why we are to expect a minimum value.

3. Find a number such that the amount by which it exceeds its square is the greatest possible.

4. Find a number such that the sum of the number and its square shall be a minimum.

5. A rectangular pasture, one side of which is bounded by a straight river, is to be fenced on the other three sides, no fence being needed on the river side. What are the dimensions of the largest pasture that can be enclosed with 1,000 rd. of fence?

6. Find the area of the largest rectangle that can be inscribed in a triangle whose base is 10 and whose altitude is 8.

7. The sum of the arms of a right triangle is 20. Find the length of each arm when the area of the triangle is a maximum.

8. The lower base of a trapezoid is 10; the sum of the altitude and the upper base is 15. Find the altitude when the area is a maximum. What is the largest possible area?

9. The perimeter of a circular sector is 10. Find the radius and the arc when the area is a maximum. How many degrees are there in the central angle?

10. For what value of k is the straight line $y = x + k$ tangent to the parabola $y = x^2 - 5x + 6$?

11. For what value of k will the two simultaneous equations

$$y = x + k$$
$$y = x^2 - 5x + 6$$

have two equal values of x?

The Inverse of Differentiation. Consider the following problem:

Given $\dfrac{dy}{dx} = 2x + 3$, express y as a function of x.

Solution. We know that the result must be of the form of $y = ax^2 + bx + c$, because when we differentiate a function of the second degree we should get one of the first degree. If it is given that

$$y = ax^2 + bx + c$$

Then, $$\frac{dy}{dx} = 2ax + b.$$

Comparing $2ax + b$ with $2x + 3$, we find $2a = 2$ and $b = 3$. The constant c disappeared in the process of differentiation and we have no way of knowing what its value is if we are given merely that $\dfrac{dy}{dx} = 2x + 3$. If we were given the additional fact that $x = 2$ when $y = 3$, we can determine c by substitution. In this case

$$y = x^2 + 3x + c$$
$$3 = 4 + 6 + c$$
$$c = -7.$$

and the final relation between y and x is given by

$$y = x^2 + 3x - 7.$$

Exercises. A few examples of the type given above will readily make the pupil able to perform the operation which is the inverse of differentiation. This inverse operation is known as integration.

A Mechanical Method of Integration. The examples of the preceding section are solved more rapidly by the following method

where it is seen that the steps of differentiation are merely reversed:

If $\dfrac{dy}{dx} = 2x^3 - 4x^2 - 3x + 2x^0$

$$y = \frac{2}{3+1}x^{3+1} - \frac{4}{2+1}x^{2+1} - \frac{3}{1+1}x^{1+1} + \frac{2}{0+1}x^{0+1} + c.$$

This method holds for all exponents except the exponent -1. It fails for the exponent -1 because the procedure then leads to division by zero, which is an impossible operation. In this case we get a logarithmic function, but we are not going to consider the case here.

Area by Integration. Integration is the inverse of differentiation and deals with the problem of finding the function when its derivative or rate of change is given. Hence the problem now is to discover the rate of change directly from the given problem and from this rate of change to find the function itself. This function is known as the *integral*.

Let us consider the area of trapezoid $OHCB$ (Fig. 9) or the area bounded by the axes, the line $y = 2x + 3$, and the perpendicular dropped from any point (x,y) on the line to the X-axis. As x and y increase, it is evident that the area will also increase. At what rate is the area A increasing? Let Δx represent the change in x, Δy the change in y, and ΔA the change in A. Then in Fig. 9 trapezoid $BCDE$ represents ΔA. Therefore

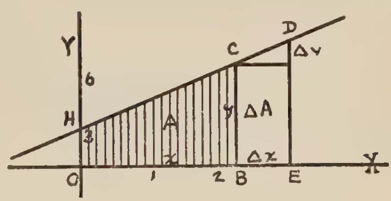

FIG. 9

$$\Delta A = \frac{\Delta x}{2}(y + y + \Delta y)$$

$$\Delta A = \Delta x\left(y + \frac{\Delta y}{2}\right)$$

$$\frac{\Delta A}{\Delta x} = y + \frac{\Delta y}{2}.$$

The value that $\dfrac{\Delta A}{\Delta x}$ assumes when ΔA and Δx approach zero is represented by $\dfrac{dA}{dx}$.

Hence $\dfrac{dA}{dx} = y$ or $\dfrac{dA}{dx} = 2x + 3$, since $y = 2x + 3$.

Integrating, $A = x^2 + 3x + c$.

But when $x = 0$, $A = 0$, because then CB coincides with HO and the altitude of the trapezoid is zero. Hence by substituting $A = 0$ and $x = 0$ in $A = x^2 + 3x + c$, we find $c = 0$ and the required area is given by the formula

$$A = x^2 + 3x,$$

a result we can readily verify by plane geometry.

To emphasize further the method of determining the constant of integration, let us determine the area in Fig. 9 above, which is bounded by the line $y = 2x + 3$, the X-axis, and the ordinates $x = 2$ and $x = 10$. It is evident that this area changes at the same rate as the one considered above. Hence

$$\frac{dA}{dx} = 2x + 3$$
$$A = x^2 + 3x + c.$$

But this area begins at $x = 2$ and A is 0 when $x = 2$. This gives us the following equation for determining c,

$$0 = 4 + 6 + c$$
$$c = -10.$$

This means that the area which begins at the ordinate $x = 2$ may be found from the equation,

$$A = x^2 + 3x - 10.$$

The area we are seeking is complete when $x = 10$,

Therefore $$\begin{aligned} A &= 10^2 + 3(10) - 10 \\ &= 120. \end{aligned}$$

This is readily verified by geometry as follows:

$$\begin{aligned} A &= \frac{1}{2}(7 + 23)8 \\ &= 120. \end{aligned}$$

The pupil should note carefully that the final answer 120 is merely the difference between two numerical values of the integral $x^2 + 3x$, namely, the value of $x^2 + 3x$ when $x = 10$ minus the value of $x^2 + 3x$ when $x = 2$. The usual way of expressing this double evaluation is $[x^2 + 3x]_2^{10}$. The numbers 10 and 2 are called respectively the upper and the lower limits of the integral. It is also evident that the constant of integration c equals the negative of the value of the integral when x equals the value of the lower limit.

By methods similar to the above we can show that the area under the curve given by $y = 6x - x^2$ may be found by integrating

$$\frac{dA}{dx} = 6x - x^2.$$

Thus,
$$A = 3x^2 - \frac{1}{3}x^3 + c.$$

The area begins at $x = 0$, hence $A = 0$ when $x = 0$, and by substitution in the last equation we obtain

$$0 = 0 - 0 + c$$
$$c = 0$$

and
$$A = 3x^2 - \frac{1}{3}x^3.$$

This gives us the area from the origin to any point (x,y) on the curve. To find the entire area above the X-axis, we must take $x = 6$, because the curve cuts the X-axis again at the point where $x = 6$. Then

$$A = 108 - 72$$
$$= 36.$$

Suppose we are required to find the area under the curve bounded by the ordinates $x = 2$ and $x = 3$. This area begins at $x = 2$, which means that $A = 0$ when $x = 2$. Substituting $A = 0$ and $x = 2$ in the equation

$$A = 3x^2 - \frac{1}{3}x^3 + c$$

We get
$$c = -\frac{28}{3}.$$

Hence the complete expression for the area under this curve which begins at $x = 2$ is

$$A = 3x^2 - \frac{1}{3}x^3 - \frac{28}{3}.$$

Since the area required is complete when $x = 3$, we substitute $x = 3$ in the expression for A just found. Hence

$$A = 27 - 9 - \frac{28}{3}$$
$$= \frac{26}{3}.$$

This is the area between the curve and the X-axis that is included between the ordinates $x = 2$ and $x = 3$.

Again we note that the final value of A is the difference between two numerical values of the integral $3x^2 - \frac{1}{3}x^3$, namely, the value of the integral when $x = 3$ minus the value of the integral when $x = 2$. The short way of expressing this difference is $\left[3x^2 - \frac{1}{3}x^3\right]_2^3$ where 3 and 2 are known as the upper and lower limits of the integral.

We also note that the constant of integration c equals the negative of the value of the integral when x equals the value of the lower limit.

The Sign of Integration. The differential of a function in x is the derivative of that function multiplied by dx. When we place the letter d in front of an expression, we indicate that the differential of that expression is to be found.

Thus $\quad d(x^2) = 2xdx, \quad d(5x^2 + 3x) = 10xdx + 3dx$

$$d(uv) = udv + vdu.$$

If
$$y = 3x^2 + 7x + c,$$
$$d(y) = d(3x^2) + d(7x) + d(cx^0)$$
$$1 \cdot y^0\, dy = 6xdx + 7x^0dx + 0 \cdot dx$$
$$dy = 6xdx + 7dx.$$

Now if we desire to indicate that the operation which we performed to obtain the differential is to be undone, we write the symbol $\int$ in front of the differential. The symbol $\int$ is called the sign of integration. It is merely an elongated S and is suggestive of the fact that integration is really a process of summation. We shall bring out this fact more fully on page 133.

Thus $\int 6xdx$ means that we are to find the function whose differential is $6xdx$. Evidently one such function is $3x^2$, but $3x^2$ is not the only such function: $3x^2 + 5$, $3x^2 - 7$, etc., all fulfill the requirement, but they are all included in the expression $3x^2 + c$, where c is an arbitrary constant. Thus we write

$$\int 6xdx = 3x^2 + c;$$

$3x^2$ is called the integral of $6x$, $6x$ is called the integrand, and c the constant of integration.

We have given on page 128 the rule for the integration of a polynomial.

Example. $\int (5x^2 - 6x + 2)\,dx = \int 5x^2dx - \int 6xdx + \int 2x^0dx$

$$= \frac{5}{3}x^3 - 3x^2 + 2x + c.$$

We may write one constant for each integration, but this is not necessary as they may be combined into one single constant.

Area Expressed as an Integral. The equation $\frac{dA}{dx} = y$, derived on page 129, may also be written

$$dA = ydx, \qquad \int dA = \int ydx, \qquad \text{or } A = \int ydx.$$

Thus to find the area under the curve $y = 6x - x^2$, we may write

$$A = \int (6x - x^2)\,dx$$
$$= 3x^2 - \frac{1}{3}x^3.$$

If we want the particular area that is situated between the ordinates $x = 2$ and $x = 3$, we have to evaluate the integral $3x^2 - \frac{1}{3}x^3$ for $x = 3$ and $x = 2$. To indicate between what values of x the area is to be taken, we place these values of x next to the integration sign. Thus the area between the ordinates $x = 2$ and $x = 3$ is indicated by writing

$$A = \int_2^3 (6x - x^2)\,dx = \left[3x^2 - \frac{1}{3}x^3\right]_2^3$$
$$= \left(3\cdot 3^2 - \frac{1}{3}3^3\right) - \left(3\cdot 2^2 - \frac{1}{3}2^3\right)$$
$$= 18 - \frac{28}{3} = \frac{26}{3}.$$

An integral with indicated limits of the variable is called a *definite integral.* An integral without such limits is called an *indefinite integral.*

Integration Is a Summation. We saw on page 121 that the area under the curve $y = f(x)$ may be obtained by first finding the sum of a series of rectangles. Thus we saw that the area given by $A_r = y_1\Delta x + y_2\Delta x + \ldots + y_n\Delta x$ approached the area under the curve when the number of divisions was indefinitely increased, or, what amounts to the same thing, when Δx approaches 0.

We have also seen that this area may be expressed by the definite integral

$$A = \int_a^b ydx.$$

This means that the troublesome summation process of page 121 may be performed by evaluating one single definite integral.

The rectangles that form part of the above summation are called the *infinitesimal elements* of the summation. By an infinitesimal we mean a variable which can be made to differ from zero by an amount just as small as we please to make it. But no constant, however small, is an infinitesimal. Thus Δx and Δy are infinitesimals in the process of both differentiation and integration (summation) because they are variables which become and remain less than any assignable amount. It is also evident that $y \cdot \Delta x$ is an infinitesimal because $y\Delta x \to 0$ when $\Delta x \to 0$, provided y is finite.

But the infinitesimal elements of the summation need not be rectangles—the process holds equally well for summing any other kind of infinitesimals. It is this very fact which makes integration such a powerful tool. For examples to illustrate this we have to refer the reader to a textbook in the calculus.

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TEACHING THRIFT THROUGH THE SCHOOL SAVINGS BANK

By CLIFFORD B. UPTON
Teachers College, Columbia University

The Importance of Thrift. A noted economist recently estimated that the income of four fifths of the people of the United States is little more than enough, even in prosperous years, to meet their ordinary expenses.[1] This means that over 93,000,000 people in this country are making only a little more than enough to provide the actual necessities of life; they can lay aside little, if anything, for a rainy day. This fact alone shows us how essential it is, for the good of individuals and the welfare of the nation, to establish the habit of wisely conserving our resources—the habit of thrift.

The Growth of School Savings Banking. One very effective means of teaching thrift, which is growing rapidly in popularity, is the school savings bank. During the past seven years, the number of schools in this country having school savings banks has increased from 2,736 to 12,678 schools; in the same period the total savings deposits in these school banks have increased from $2,800,000 to $24,000,000. Official reports show that during the school year 1926-27 there were enrolled in these 12,678 schools 4,658,000 pupils, of whom 3,815,785 actually participated in school savings.

The school savings movement has developed most rapidly in towns and cities. It has had relatively small growth in the rural districts because of the lack of contact with banks. The progress of school savings banking is generally forward in any state where once it has gained a considerable foothold, and in certain states it has assumed really large proportions.[2] During the past fourteen years the Savings Bank Division of the American Bankers Associa-

[1] From a report, made in November, 1927, by Professor Irving Fisher of Yale University.

[2] Great progress has been made in California, Connecticut, Illinois, Indiana, Massachusetts, Michigan, Minnesota, New Jersey, New York, Ohio, Pennsylvania, Rhode Island, Washington, and Wisconsin.

tion has been active in encouraging school savings and in providing schools with practical guidance concerning the installation and successful operation of school savings banks. The most authoritative book on the subject, entitled *School Savings Banking* (published by the Ronald Press, New York), has also been prepared under the auspices of the American Bankers Association.[3] While school savings banking in its earlier stages was developed largely by bankers and business men, educators are now beginning to appreciate its value as a most efficient means of teaching the habit of thrift to school children.

The Schools Must Teach Thrift. It may be contended that thrift can better be learned after the child leaves school and begins to earn money regularly. But both experience and the principles of teaching show us that then it is too late. The teaching of thrift is particularly the task of the school. This does not mean that bankers should abandon their interest in the school savings bank; it means, rather, that bankers and educators should coöperate very closely in making the school the medium for inculcating the thrift habit.

There are several outstanding reasons why the school is the place for the teaching of this valuable habit. In the first place, we know that habits are most easily formed in childhood. And the school commands the child when he is young. In the second place, thrift is a very broad and complex idea, embracing not only the wise saving and spending of money but also the efficient use of all our personal and national resources.

There are many opportunities in the school for teaching this larger notion of thrift. In the arithmetic class, for example, those lessons in thrift can be taught that have to do with the proper saving and investing of money. In the geography class those phases can be presented that are concerned with the resources of our country and their economical use; in the hygiene and home economics classes instruction can be given concerning the advantages that come through good health and the proper selection and conservation of food; while in the science classes the conservation of human energy and other economies that have come through the modern application of science to industry can be presented. Thus the notion of thrift can be introduced in many school subjects.

[3] Teachers interested in school savings banking will find it worth while to write to the Savings Bank Division of the American Bankers Association, 110 East 42nd Street, New York City, for such literature and reports as they issue.

Thrift Must Become a Habit. But thrift, to be taught most effectively, must be made a habit which will work automatically like all other habits. The school has as one of its primary tasks the forming of habits in children. Reading, writing, and computing are complex habits that the school forms by instruction of a gradually progressive kind in which repetition plays an important part. Similarly, eating, walking, speaking—all these are habits. Even a person's thoughts concerning his neighbors or the government may be largely matters of habit. Practically all the things we do easily and automatically are habits. Thrift, therefore, should be made a habit with the child if we wish him to respond quickly and persistently to it. Since it takes a long time for the child to acquire a habit, the school must be depended upon to surround him with the necessary influences and repetitions until the notion and the practice of thrift become habitual with him.

No habit, whatever it may be, can be successfully taught to anyone by simply talking about it. All habits, to be effective in life, must be reduced to action. You may talk about handwriting, and the crossing of the "t," and the dotting of the "i" as long as you wish, but that does not mean that the child will necessarily cross the "t" and dot the "i." You have to establish the habit of crossing the "t" by having the child actually do it over and over again until he can do it easily, without effort, and without thought. It is the same with the habit of thrift.

Thrift has always been talked about in the schools. It is talked about today more than ever before. The school savings bank offers something better than talk; it offers the daily opportunity to practice thrift, the daily example of others practicing thrift, and a vitally interesting and stimulating introductory contact with the world of real business.

The School Savings Bank Utilizes the Laws of Learning. For the purpose of teaching the child the habit of thrift, the school savings bank is a psychologically sound medium and one of the best now available. Any sort of learning proceeds through five fundamental laws which have been formulated by psychologists. All effort to promote thrift, among either adults or children, including advertising and drives as well as more subtle propaganda, should be in accordance with these fundamental laws if it is to be thoroughly effective. Much thrift promotion that has been undertaken by banks and schools has been a waste of energy because it

has not been followed up by a well managed school savings system. For the school savings bank, as you shall see below, puts into practice, if it is properly handled, all the fundamental laws of learning.

First Law of Learning. The first law of learning points out that to inculcate any habit you must first create on the part of the pupil the *desire* to acquire that habit. The school savings bank accomplishes this in respect to the habit of thrift. Posters, stories, talks, and subtler suggestions about saving for a time of need may interest a boy vaguely in thrift, but nothing stimulates him to actual saving so much as the presence of a real bank in his school and the sight of his classmates investing their pennies and nickels and increasing the totals in their pass books. And when he finally finds his bank account big enough to buy the roller skates he has always wanted, he has learned a lesson that will be a stimulus to further thrift.

Second Law of Learning. The second law of learning states that to teach a habit you must supply a certain amount of *fundamental knowledge* concerning the habit to be formed. You do not want a child to form a habit blindly, in a purely imitative way. He should know what he is doing. One important item of information which the child gets from the school bank is a first-hand demonstration of how to deposit money in a savings bank. He also learns that his money is much safer in a savings bank than it would be at home in a toy bank or hidden away in a box. He finds that his pass book gives him a definite record of the growth of his savings from month to month. When interest is credited to his account, he begins to realize that his money is working for him and that a new force is actively increasing his total fund. A few years later he commences to understand the great power of compound interest in making his savings grow still more rapidly.

Third Law of Learning. The third law of learning states that in acquiring a new habit there must be sufficient *repetition.* No one forms a habit by keeping at a thing for a few months and then stopping. In fixing a habit the element of repetition, the doing of a thing over and over again, is the important factor. The school bank keeps the idea of saving constantly before the child, until it becomes second nature for him to think about saving and to act promptly upon that thought by the regular investing of small sums. Even on those days when he has no money to deposit, the

child continues to be reminded of the importance of saving as he sees his classmates forming in line to put their savings in the school bank.

Fourth Law of Learning. The fourth law of learning reminds us that there must be provision for the *practice* of the new idea or habit. It is not sufficient for the child to learn the theory of savings banks in school and to be told that he should use those banks when he grows up. A savings bank on hand in the school encourages the child to act upon every impulse toward thrift which may be aroused in him. Every time he deposits money in the school bank, he is strengthening the habit of thrift.

Fifth Law of Learning. And the fifth law of learning indicates that some sort of *satisfaction* must result from acquiring the habit. A man who has worked all his life to save enough to put his son or daughter through college will have sufficient satisfaction if that son or daughter is successful in college. And so with every habit that is taught we must see that there is some resulting satisfaction rather than annoyance or disappointment if we wish the habit to endure. The school savings bank offers a child the satisfaction of seeing his possessions grow through repeated deposits and the accumulation of interest. He may use the money thus acquired to buy something he particularly desires or he may keep it in the bank as a basis for still larger savings. Either use will give the child pleasure and so increase his interest in thrift.

Thus the school savings bank is in every way an efficient means of teaching the habit of thrift, which is a habit of such vital importance to young Americans of the present day.

Coöperation with Bankers Essential. In the further development of the school savings movement in this country, it is very essential for educators to do everything possible to assure the continued coöperation and support of the bankers. It is only because they have recognized the importance of encouraging thrift that bankers have done so much to promote school savings banks. Bankers do not derive any financial gain from the deposits made by the school children; on the contrary, the expense of promoting and installing school banks, of keeping the children's accounts, and of paying interest on such small deposits, is so great that the school savings systems usually entail a financial loss to the banks that are handling them. But the bankers have been willing to risk this business loss, just as the teachers have been willing to expend

extra time to make this movement a success, because they feel that the school savings bank will help the boys and girls of today to become thrifty men and women.

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MEASUREMENT AND COMPUTATION

By GEORGE W. FINLEY
State Teachers College, Greeley, Colorado

Two Maxims. Two maxims are coming to be recognized as fundamental in the modern teaching of mathematics: "Teach the children to use common sense in handling numbers." "Develop a genuine number sense in connection with elementary mathematics." If Mary is trying to find out how much 6½ yards of cloth will cost at 55 cents a yard, she is expected to know not only that she should multiply and how to multiply, but that her result should be somewhere near 3 or 4 dollars. At least she should know that if she gets $35.75 there must be something wrong. She makes a rough estimate and checks her computation against it.

In keeping with this movement progressive teachers have sought to teach children how to handle computations from actual measurements in a common-sense fashion. If they measure their room and find it to be 20.2 feet by 24.8 feet, they should know better than to say that the area of the floor is 500.96 square feet. Or if they find the diameter of a circle to be 6.1 inches, they should not say the circumference is 19.16376 inches.

Some teachers feel that this whole subject is so complicated and difficult that it cannot be successfully taught to children below the senior high school. Experience has shown that this is not the case. The subject has been taught in the junior high school and has been found to be as interesting and as capable of being mastered as most of the other topics we deal with in these grades.

The two things that we need most in teaching measurement and computation are, first, an adequate treatment in our textbooks and, second, a clear understanding of the subject, together with a realization of the specific objectives to be reached. The first of these needs is now being met by some of our textbook writers. The second will be met as soon as teachers as a whole recognize the importance of the subject. I shall set forth the essentials in what follows.

I. Approximate Measurement

Measurement Approximate. Every measurement is an approximation. When we say that line AB is 8.3 inches long, we mean that when A is at the zero-point of our ruler B is nearer to 8.3 than it is to 8.2 or 8.4. The line therefore lies between 8.25 inches and 8.35 inches. If we say that line AB is 18 inches long, we mean it lies between 17.5 inches and 18.5 inches. When we say that the distance to the moon is 240,000 miles, we mean that that distance is between 235,000 and 245,000 miles. Every measurement, then, is accurate only to a certain degree. To understand how this is described we need to understand the expression "significant figures."

Significant Figures. There are two ways of describing the accuracy of our measurement. One is to say that it is correct to units, tenths, hundredths, and so on. Thus 28 inches is correct to units, 28.4 inches to tenths, and 28.42 inches to hundredths. A better way is to say that the first of these measurements shows two-figure accuracy, the second three-figure accuracy, and the third four-figure accuracy. When we say two-figure accuracy we mean two "significant figures." The 240,000 used in the preceding paragraph indicates two-figure accuracy; for although it contains six figures, only two of them are significant. In 4.0 inches we have two-figure accuracy because the zero indicates that we have measured to the nearest tenth.

The best way to make this clear to children is to give them numerous illustrations and have them tell the number of significant figures in each: 34 has two significant figures; 6.3 two; 304 three; 8,004 four; 8.0 two; 0.40 two; 9.00 three; 0.004 one; 0.0040 two; and 93,000,000 two. We may make the following statements about significant figures:

1. All the digits except zero are significant wherever used.
2. Zeros are always significant when they occur between other digits or at the right of a decimal, as in 403 and in 4.60.
3. Zeros are never significant when there are no other digits to the left of them, as in 0.02 and in 0.0024.
4. In general zeros which occur at the right of a whole number are not significant. When we say the distance light travels in a second is 186,000 miles we are using three significant figures. Sometimes, however, such zeros are significant. If we say the distance between two towns is 10 miles we usually mean that the distance is between

9.5 and 10.5 miles. This makes the zero on the 10 significant. In such cases we can usually tell from the nature of the measurement whether or not the zeros are significant.

Rounding off Numbers. If we are to use common sense in computing with approximate numbers, we need to know how many figures we should use and how many figures we should retain in the result.

If Jack measures the diameter of a circle and finds it to be 8.4 inches, what value of pi should he use to get the circumference? Should he use 3.14159? As we will show later he should drop off all the figures to the right of the 4. Then when he gets his answer he should drop off all but two of the figures. This "dropping off" of figures we call "rounding off" the number. The usual rule followed here is this: *When the digit dropped is 5 or more, increase the preceding digit; when it is less than 5, retain the preceding digit unchanged.* For more refined work, in statistics for instance, this rule is modified to: *When 5 is dropped, increase the preceding digit if it is odd and leave it unchanged if it is even.*

Just how much and when numbers should be rounded off will appear in the following paragraphs.

II. Computation with Approximate Numbers

Addition and Subtraction. In order to make clear how certain computations with approximate numbers may be carried on, let us take a typical example from the mathematics classroom. Suppose four children have been asked to measure the lengths of the four walls of their room and they turn in the following measurements: 34 ft.; 42.2 ft., 34.15 ft., and 42.155 ft. What should the class do with these numbers to find the distance around the room?

Of course one of the important lessons to be taught here is that measurements which are to be added or subtracted should all be carried out to the same place: units, tenths, hundredths, or thousandths. But since we already have these measurements we can use them to find the distance around the room correct to units.

The children set down the least possible value of each measure-

33.5	ment as shown at the left. In a similar way	34.5
42.15	they set down the greatest possible value of	42.25
34.145	each measurement as shown at the right.	34.155
42.1545	The distance around the room, then is some-	42.1555
151.9495	where between 151.9495 and 153.0605.	153.0605

Now have the children add the numbers as they were given, as shown on the left, and add the rounded off numbers as shown on the right. It is clear that since one wall was measured to units only we cannot depend upon any digits to the right of this place; also that the sensible thing to do is to round off our numbers before we add. It is well for us to remember to keep one digit more than we shall use in the final result because of the carry number.

$$\begin{array}{r} 34. \\ 42.2 \\ 34.15 \\ 42.155 \\ \hline 152.505 \\ \text{or } 153 \end{array} \qquad \begin{array}{r} 34. \\ 42.2 \\ 34.2 \\ 42.2 \\ \hline 152.6 \\ \text{or } 153 \end{array}$$

Rule: *To add or subtract approximate numbers whose final digits do not fall in the same column, round off each number to within one of the place occupied by the final significant digit which falls farthest to the left; add or subtract and round off another figure in the result.*

Multiplication. Suppose the children set out to find the area of the floor of their room from the following measurements: width 34 ft., length 42.155 ft. They first find the least possible area as shown on the left and then the greatest area as shown on the right. They see that the only thing they know about the area is that it is between 1412.17575 and 1454.36475. All the digits are quite uncertain except the two on the left.

$$\begin{array}{r} 42.1545 \\ 33.5 \\ \hline 2107725 \\ 1264635 \\ 1264635 \\ \hline 1412.17575 \end{array} \qquad \begin{array}{r} 42.1555 \\ 34.5 \\ \hline 2107775 \\ 1686220 \\ 1264665 \\ \hline 1454.36475 \end{array}$$

Now if they round off the 42.155 to within one of the number of figures in 34 (the less accurate factor) and multiply 42.2 by 34, they get 1434.8, which is well within the original limits.

Take another example. If the diameter of a circle is 8.4 ft., what is its circumference?

Since 8.4 has two-figure accuracy, we will use 3.14 for the value of pi. Then $3.14 \times 8.4 = 26.376$, which rounds off to 26.4. The least possible value in this case is 26.232 and the greatest 26.546. Again our value of 26.4 is well within the limits.

We could round off the more accurate number to the same number of figures given in the less accurate number but this would sometimes give us a result outside the desired limits. Take 3.162×9.4. The least value of the product is 29.560 and the greatest 29.885. But $3.2 \times 9.4 = 30.08$, which is more than the greatest possible value. It is therefore better to carry one extra figure on the more accurate factor.

Rule: *To multiply approximate numbers round off the more accurate factor to within one of the number of significant figures in the less accurate factor, and multiply and round off the product to the number of figures in the less accurate factor.*

A More Accurate Method. After a class understands the use of literal numbers an interesting exercise can be worked out by showing how to find just what the error will be when any two approximate numbers are multiplied. To do this we must first understand the difference between absolute error and relative error.

If we say a distance is 12 in., the greatest possible absolute error is ½ of 1 in. or 0.5 in. The relative error is 0.5 in. in 12 in. or $\frac{0.5}{12}=\frac{1}{24}$, or about 4%. Again, the absolute error in 93,000,000 mi. is 500,000 mi. The relative error is $\frac{500{,}000}{93{,}000{,}000}=\frac{5}{930}$ or about 0.5%. We see, then, that the relative error is the absolute error divided by the obtained measurement.

Now suppose we are to multiply a by b. Let the absolute error in a be e_1, and in b, e_2. Then the actual values of a and b are $a + e_1$ and $b + e_2$, where e_1 and e_2 may be either positive or negative. Then we have the relative error in $a = \frac{e_1}{a}$; and the relative error in $b = \frac{e_2}{b}$. The product of the actual values of a and b is $ab + ae_2 + be_1 + e_1e_2$. But e_1 and e_2 are both very small compared to a and b. Hence their product will be so small as to be of no importance. Dropping this term we have the product error $ae_2 + be_1$. Then the relative error in the product is $\frac{ae_2 + be_1}{ab}=\frac{e_2}{b}+\frac{e_1}{a}$. We see, then, that the relative error in the product is equal to the sum of the relative errors in the two factors.

Now let us apply this principle to the problem of finding the error in the product of two approximate numbers. Take 342 × 3.1.

$a = 342$ and $e_1 = 0.5$ $\qquad \frac{e_1}{a}=\frac{0.5}{342}=0.0014$

$b = 3.1$ and $e_2 = 0.05$ $\qquad \frac{e_2}{b}=\frac{0.05}{3.1}=0.0161$

$$\begin{array}{r} 342 \\ \underline{1026} \\ 1060.2 \end{array}$$

Product relative error = 0.0175

$$1060.2 \times 0.0175 = 18.55, \text{ total product error}$$

$$1060.2 - 18.55 = 1041.65 \qquad 1060.2 + 18.55 = 1078.75$$

Hence our product is somewhere between 1041.65 and 1078.75. In other words, it is 1060.2, with a possible error of 18.55 either way. This is written thus: 1060.2 ± 18.55.

Division. Since the product of the quotient and the divisor equals the dividend, our rule for division of approximate numbers will naturally follow the rule for multiplication. Let us find the diameter of a tree whose circumference measures 12.8. Since this number has three-figure accuracy, we will use 3.142 as our value of pi.

$$12.8 \div 3.142 = 4.073 \text{ or } 4.07$$

Rule: *To divide approximate numbers round off the more accurate number to within one of the number of significant figures in the less accurate number, divide and carry the quotient out to the number of figures retained in the more accurate number, and then round off the last digit.*

The short methods sometimes given for multiplying and dividing, by means of which we can leave off the extra figures in the partial products, are of doubtful value. The danger of additional errors offsets any value that might come from shortening the processes involved.

Exact and Approximate Numbers. We must be careful to see that children know when these rules apply. When we multiply two approximate numbers, do we always round off?

If 350 children give 5 cents apiece toward a library fund, how much will they have in the fund? The result is evidently $17.50 and there is no rounding off. The point is, of course, that these are not approximate numbers. They arose from an actual count of objects. There were exactly 350 children and each one gave exactly 5 cents. We therefore call such numbers exact numbers.

It is clear, then, that we get exact numbers when we count objects and approximate numbers when we measure, or when we round off such numbers as pi or the square root of 2.

Accuracy in Measuring. How accurately should we measure? Well, of course, that all depends. It depends upon at least three things:

1. Our measuring instruments.
2. The distance to be measured.
3. The purpose of the measurement.

If we are to measure the distance across a room we can get a much more accurate result with a steel tape than with a foot-rule or a

yardstick. We can measure the diameter of a bolt much more accurately with a micrometer than with an ordinary ruler. If we measure the width of a desk with a yardstick showing inches and quarters of an inch, we should be able to get a result accurate to the nearest eighth of an inch, one-half of the smallest division on the yardstick. If we use a meter stick showing centimeters and fifths of a centimeter, we should measure to the nearest tenth of a centimeter, one millimeter. Thus a single measurement cannot in general be more accurate than one-half of the smallest division on the measuring instrument used.

But if the object to be measured is longer than the measuring stick used, there is another source of error. Each time we pick up our stick and lay it down again we make a slight error. How much this will be depends upon the care with which we measure. When we measure a board, say six feet long, with a foot-rule, and mark the end of the rule with a dull pencil, we may miss the real length of the board by five times the width of the pencil mark. If we use a sharp pencil, or better still, a sharp knife, our measurement will be more accurate. The total amount of error, then, will depend upon the length of the line to be measured.

In stating the degree of accuracy required in any kind of measurement we often give, not the total amount of error, but the ratio of the total error to the distance measured. If in measuring the top of a desk one gives the result as $28\frac{1}{4}$ inches, we understand him to mean that he has measured to the nearest fourth of an inch. The possible error, therefore, is $\frac{1}{8}$ of an inch. An error of $\frac{1}{8}$ in $28\frac{1}{4}$ or $113/4$ is $\frac{1}{8}$ in $226/8$ or an inch in 226 inches. If we say the average distance from the earth to the sun is 93,000,000 miles, our error may be 500,000 miles; this is an error of 5 in 930 or 1 in 186. Two-figure accuracy will give an error of 10% or less, three-figure accuracy an error of 1% or less, and four-figure accuracy an error of 0.1% or less. This fact will enable us to tell how many places we should have in our tables for such work as that found in numerical trigonometry.

The most important thing in determining the degree of accuracy for any given measurement is the purpose of the measurement. If we are laying off a ball diamond at a picnic we would not mind an error of 1 in 20 or even more. But if that diamond is being laid off in a new field for one of our big league teams an error of 1 in 1,000 would be considered entirely too great.

Surveyors suit the degree of accuracy to the purpose of the survey. When they are surveying in the country where land is comparatively cheap they are satisfied with an error of not more than 1 in 300. In a large city, however, where land is very valuable, they cut the error down to 1 in 50,000.

Summary. Just what are the specific things we need to teach, then, in order to enable children to use approximate numbers in a sensible fashion? They are:

1. A realization of the fact that all numbers derived from measurement are approximate.
2. The meaning of significant figures.
3. How to round off numbers to any desired degree of accuracy.
4. How to add and subtract approximate numbers.
5. How to multiply and divide approximate numbers.
6. Which numbers are approximate and which are exact.
7. Some knowledge of the degree of accuracy that should be attained in any given measurement.

These things can be taught to children of junior high school age in such a way that they will be understandable, interesting, and worth while.

THE TEACHING OF DIRECT MEASUREMENT IN THE JUNIOR HIGH SCHOOL *

By WILLIAM BETZ

East High School and Junior High Schools, Rochester, N. Y.

I. Introductory Statement

Counting and Measuring. Mathematics has been defined as the science of measurement. This definition, while very incomplete, may serve to remind us of the well known fact that the two processes which caused mathematics to come into existence, at the dawn of human history, were counting and measuring. In other words, mathematics has a dual foundation which, to this day, has remained the chief justification for including mathematical training in the education of every normal child.

Some of the Aspects of Measurement. In the following pages it is proposed to examine briefly some of the aspects of *measurement,* in so far as it constitutes an essential and recognized ingredient of the mathematical curriculum of the junior high school. It is hardly necessary to justify such an investigation beyond calling attention to the prevalent confusion of educational objectives,[1] to the resulting uncertainty with which teachers are facing even urgently needed readjustments, and to the lack of agreement which is revealed by recent textbooks and courses of study.[2] In the mathematical curriculum of the junior high school, measurement, as a separate "topic" or "activity," is usually listed under

* This report was prepared in connection with the Rochester Coöperative School Survey.

[1] See, for example, Bode, B. H., *Modern Educational Theories,* Macmillan Co., 1927; Davis, C. O., *Our Evolving High School Curriculum,* Chap. VI, World Book Co., 1927; Monroe, W. S., *Directing Learning in the High School,* Chap. III, Doubleday, Page, and Co., 1927; Briggs, Thomas H., *Curriculum Problems,* Macmillan Co., 1926; Stratemeyer and Bruner, *Rating Elementary School Courses of Study,* Bureau of Publications, Teachers College, Columbia University, 1926.

[2] See the *Fifth Yearbook of the Department of Superintendence,* Chap. XI, Washington, 1927, containing the report of a special committee on Junior High School Mathematics; article on the "Development of Mathematics in the Junior High School," in the *First Yearbook of the National Council of Teachers of Mathematics;* "Mathematics and Curriculum Tendencies in Secondary Education," by V. T. Thayer, *Educational Administration and Supervision,* September, 1927.

the general heading of Intuitive Geometry. Thus, the National Committee on Mathematical Requirements included the following items in its tentative formulation of the content of Intuitive Geometry:

(a) The *direct measurement of distances* and angles by means of a linear scale and protractor. The approximate character of measurement. An understanding of what is meant by the degree of precision as expressed by the number of "significant" figures.

(b) Area of the square, rectangle, parallelogram, triangle, and trapezoid; circumference and area of a circle; surfaces and volumes of solids of corresponding importance; the construction of the corresponding formulas.

(c) Practice in *numerical computation* with due regard to the number of figures used or retained.

(d) *Indirect measurement* by means of *drawings to scale;* use of *square ruled paper.*

Now, the mensuration of the common plane figures and of the most important solids has long been treated more or less adequately in the standard textbooks on arithmetic. Moreover, the subject of "weights and measures," very naturally, is receiving constant attention in the elementary grades. This is not true, however, of either direct or indirect measurement. Emphasis on these skills and abilities varies enormously, ranging all the way from a casual reference to an extensive and highly specialized treatment. Is it possible to suggest a sane middle ground which is feasible in the average classroom?

Measurement of Line-Segments. For reasons of economy, only the case of *direct* measurement will be fully discussed. Again, since the technique employed in direct *linear* measurement may easily be adapted by any intelligent teacher to all the other phases of direct and indirect measurement, we shall consider only the measurement of line-segments. The following questions, therefore, seem pertinent in connection with this study:

1. What is the cultural background that constitutes a compelling motivation for direct linear measurement?
2. Is there a tested classroom technique which can safely be recommended to the average teacher?
3. What additional professional training can be suggested for the teacher in connection with this work?

In the three sections which follow, these questions will be considered in succession. The discussion will necessarily be frag-

mentary. It will be based, however, on extensive classroom experience, and will avoid merely theoretical opinions. In spite of the great and universally recognized importance of measurement, the existing literature on the subject usually is either too technical or too specialized to be of practical value in the schoolroom. It is to be hoped that these pages may induce others to take a more active interest in the development of reliable classroom procedures and in the creation of a more extensive pedagogic literature on the subject.

Part One

The Cultural Background of Measurement

Motivation. A convincing and sufficient motivation for this subject can be found, first, by stressing the all-pervading significance of measurement in everyday life, and second, by introducing the story of the gradual evolution of standard units as a necessary by-product of a developing civilization.

I. The Importance of Measurement

The "Master Art" of Measurement. Perhaps the most eloquent utterance in recent years on the rôle of measurement in modifying and reconstructing human life has come from the authoritative pen of Henry D. Hubbard, Secretary of the United States Bureau of Standards. In an address [3] at Lake Placid on July 10, 1925, Mr. Hubbard discussed the "master art" of measurement in language that has in it the quality of poetry.

Measurement is the Master Art. Measures tell how Nature behaves and how to control her mighty forces in the service of man. Through measures of sun and moon, their place, mass, and motion, we predict the tides on which commerce rides with her argosies to all the world. Men go by land, sea, and air, night and day, unceasingly guided by measuring instruments which tell the tale of speed, place, direction and power. Transit is "made to measure."

Explorer and surveyor locate position by measuring the height of the stars. The map is a fabric woven of latitudes and longitudes. On it the facile pencil of the engineer creates a new earth. His dreams come true because the workers, from survey to steam roller, work true to measure.

The mariner still hitches his wagon to a star, for measurement was born among the stars. The star-gazing dreamers of yesterday gave us astronomy, chronometry, the calendar, surveying, geometry, and the art of navigation.

[3] Reprinted in the official organ of the Metric Association, the quarterly journal *Measurement*, October, 1925.

Modern science began with measurement. We measure the rock to re-create the time table of geology. We measure tree rings to learn the life story of the tree, and through measures these tongues in ancient trees re-tell climatic history centuries past.

To science, *measurement is a means of discovery, prophecy, control.* To industry, it is the tool of creation. The measured curves of every tool are alert with the skill of a race of craftsmen. The machine is a complex of measures which sets each craft to cosmic power.

All industry measures to serve. Its every deed fits a measured need, whether size, strength, color, or whatever gives utility to things. *Industry is service set to measure.* We measure the body to clothe it with measured apparel. Our life itself fits into measured schedules of time and place.

Measures are the life of the fine arts—poetry sings in measures, sculpture carves them into inspirations, architecture enshrines them, and the measured record of tones and silence, their sequence and concord, transmutes the soul of the master musician into vibrant harmonies for the joy of the world.

Measurement is a miracle worker. We give a measured curve to glass to match a measured defect of the eye, and restore sight to the aged and perfect the vision of youth.

Perfect measurement is perfect truth, and sets man free. Everywhere measurement is busy creating the tomorrow of our dreams. Measurement is the master art. It has but one purpose, to create the maximum happiness for all.

At the Philadelphia meeting of the Metric Association, December 27, 1926, Mr. Hubbard amplified these statements,[4] in equally inspiring phrases, as follows:

Protagoras begins one of his books by saying "Man is the measure of all things." Surely man is the measurer, and measurement his master art. Measurement is not merely *an* art, it runs through *all* arts, sciences, industries—it is *the* master art.

Measurement is a pioneer. Early history writ on trees marked by notches the height of flood and the passing days. Man measured the earth, geometry; the turning shadows of the day, chronometry; the stars and their motions, astronomy; the seasonal migrations of the sun, chronology. Early life depended upon measuring the turn of the year, when the sun starts north and seedtime nears. The pyramids of the Pharaohs and the Lat of Asoka were vast sundials of the seasons whose noon shadows fixed the best time of planting to assure the needed crops on which wealth and even survival depended.

Through measures alchemy was transmuted into chemistry, magic into physics, and astrology into astronomy. Measures bring new eras. The theodolite precedes the railway; the micrometer ushers in the machine tool age; the pyrometer, a new era in metals; the kymograph, a new psychology; the spectroscope, a new knowledge of atoms.

[4] See the January, 1927, number of *Measurement.*

Here is a shoe factory with a houseful of lasts. The shoe last sums up centuries of shoemaking art in a set of measures—length, width, instep, ankle,—by which the shoe is built, classified, sold, and worn.

A dress pattern sums up an age-old art—clothing the body. The pattern is a complex of measures. Artists of the mode build their creations on measures of the body. Every cut of the shears or stitch of the needle is measured to ensure perfect fitting for comfort, taste, or health.

Feeding the race, a primary need, creates countless recipes which set to measure the skill of the cook, and thus make reproducible a host of delectable dishes.

What is true of the shoe last, the dress pattern, and the recipe is true of a hundred thousand products of industry. If we need a hat, gloves, collars, the first question is size. The machine knows only the measures, the user only the quality.

The rôle of measurement in medical research and practice would make a fascinating volume. By measuring the area of a gunshot wound Carrel can foretell the day on which healing will be complete. So accurate is the prediction that it can be used to test the value of differing treatments and to rate the reagents used for asepsis. The white rat, probably the best measured animal, is being used to measure vital facts and factors of life itself, and the results are being used to perfect human life.

Every new kind of measurement adds new perceptive powers to our five senses, and even creates new senses of perception. We cannot feel the turning of the earth nor its swing around the sun, but the gyrostat can detect it for us. Starlight shining on a weld of two metals causes an electric current which turns a mirror. The angle measures the radiant energy. A million years of direct shining of the Polar star would barely heat up a millimeter of water one degree if no heat escaped.

Such infinitesimals we measure daily and nightly in our laboratories and observatories, for our measuring instruments detect a thousand things to which our senses are insensitive.

. . . . .

By measures we can predict. Measured correlations give us the gift of prophecy. Measurement assures us that nothing is fortuitous, for plotted curves of almost every phenomenon run parallel to others and thus two series of measurements seemingly remote reveal causal or sequential relations which follow *simple laws.*

As measured mechanism delivers automatically a thousand kinds of measured service each gaged to perfectly satisfy some need or desire, life itself becomes freer, less mechanical. We speed up travel threefold in a decade, leave road and rail behind to dive the trackless sea or mount the thin air. Clothes make the whole world habitable, shoes carpet the earth with leather, planes conquer altitudes and radio nullifies distance. Fire turns winter into spring, devices give us ice in the summer. A tungsten thread turns night to day. A copper wire makes the world a whispering gallery, or, coiled in a motor, sets free the toiling hand of man.

The blue print speaks the language of measurement. It is a congeries of

measures, each helping to creat the new mechanical marvel. The blue print is the measured chart of new utopias. In study, shop, and laboratory the world over "Tomorrow" is being traced in paper dreams set to measured scales. Measurements are thus shaping the pattern of the "Wonderlands of Tomorrow."

Man's miracles multiply, break the chain of time, place, and circumstance to give him vast degrees of freedom and new and limitless powers. Little wonder that Emerson, facing the new age of science and its possibilities, declared "I have never known a man as rich as all men ought to be."

Of course, it is utterly impossible to place before a pupil in the seventh grade the imposing array of illustrations listed above, nor is it necessary to recite such an exhaustive catalogue of facts. The teacher should, however, possess this reserve fund of information, and should dispense it to her pupils at such times and in such quantities as may best meet the actual needs of the moment. It is sufficient, at the beginning, to depend upon two very elementary arguments on behalf of accurate measurement. These arguments never fail to convince even that large class of pupils who are described by teachers as "slow," "lazy," "dull," "having no interest in mathematics," and the like. In simple language, these arguments may be stated as follows:

1. We must learn to measure accurately in order that "things may fit."
2. Without accurate measurement, there would be much waste of material, time, money, and energy.

II. The Story of Measurement

The Need of an Adequate History. A brief and yet authentic history of measurement, simple enough for classroom purposes, has not yet been written. The original source books are to be found only in large libraries, and the technical treatises on metrology make their appeal only to specialists. We need condensed and readable accounts comparable to those which have told the story of counting and of our number system.[5] The following fragmentary notes are offered in the hope that they may soon be replaced by an adequate summary.

The Evolution of Standard Units. The story of measurement is of absorbing interest. The gradual evolution of linear measure-

[5] See, for example, Professor D. E. Smith's admirable book, *Number Stories of Long Ago*, Ginn and Co., 1919.

ment may be presented essentially as a drama of three acts. Act I tells of the *human body* as the primary source of reference, and of the resulting age-long uncertainty and confusion. Act II presents the struggle to obtain a more permanent standard, through the substitution of the *earth* as a standard of reference. Act III shows the attempt to attain an absolute standard by a dependence on the immutable laws of nature, substituting for the terrestrial unit a *cosmic* unit.

For the origin of weights and measures we must go back to the earliest days of the human race. The idea of measuring arose almost simultaneously with that of number. The nomadic life of primitive man constantly suggested the measurement of *itinerary* distance. Thus the distance traveled in one day would become a natural measure of journeys. For shorter distances, the *pace* would be taken as the unit, or even the foot. As soon as a more stable type of social life developed, the activities centering around food, clothing, shelter, the making of weapons, household implements, and the like, led spontaneously to constantly recurring *manipulative* measurements. Even primitive man would thus be led to employ as linear measures also the breadth of a finger, the breadth of the hand, the span of the extended fingers of one hand, the length of the forearm, and the distance between the tips of the fingers when the arms were outstretched. In other words, the *human body* furnished the earliest standards of length. This was both fortunate and unfortunate. It was a help in giving a universal background to the processes and the terminology of measurement, reflected to this day in the use of such terms as foot, mile, and hand. It was a source of confusion, since it created a multiplicity of fundamental standards of reference which, as we are all aware, has never completely disappeared.

This diversity of purely subjective units of length soon proved embarrassing for purposes of trade and commerce. A desire for *uniformity* arose. It was seen that the interests of all would best be served if a single unit should be employed throughout the tribe. This single standard might be derived by averaging an arbitrary number of the units in actual use, or by having some standard imposed by the authority of the ruler of the tribe. The latter custom was practiced very commonly. Tradition reports that as late as the time of Henry I (1100), the length of the English yard was fixed by the length of the king's arm. But still there was confusion. Each

country, and often each township, would boast of its own system of weights and measures. And so the demand for uniformity was bound to continue. That a solution was not found until the arrival of our own era was due, first, to the political turmoil constantly prevailing in nearly all parts of Europe, and second, to the lack of an adequate scientific basis. The Renaissance and the subsequent revolution in all branches of human learning at last prepared the way for the creation of the necessary theories and the corresponding scientific equipment. A group of great thinkers arose who ushered in the modern conception of the world in which we are living. It was their influence which led to the substitution of the *earth* as a standard of reference. But it still required the throes of the French Revolution to bring this dream to practical realization through the creation of the metric system.

And yet, even the meter, in spite of the utmost care used in the original measurements, was seen to be theoretically imperfect. The earth is not a perfect sphere, and human instruments always introduce sources of error. Again science proposed a more ideal solution. When Professor Michelson succeeded in defining the meter in terms of light waves, the standard unit of length had at last been emancipated, and we now have a "meter" as permanent as the laws of nature.

The Human Body as a Standard. The development of the manipulative units based on the human body is well described in a classic American treatise on weights and measures, written by John Quincy Adams as Secretary of State of the United States, in the following passage:

> The proportions of the human body, and of its members, are in other than decimal numbers. The first unit of measures, for the use of the hand, is the *cubit,* or extent from the tip of the elbow to the end of the middle finger; the motives for choosing which, are, that it presents more definite terminations at both ends than any of the other superior limbs, and gives a measure easily handled and carried about the person. By doubling this measure is given the *ell,* or arm, including the hand, and half the width of the body, to the middle of the breast; and, by doubling that, the *fathom,* or extent from the extremity of one middle finger to that of the other, with expanded arms, an exact equivalent to the stature of man, or extension from the crown of the head to the sole of the foot. For subdivisions and smaller measures the *span* is found equal to half the cubit, the *palm* to one-third of the span, and the *finger* to one-fourth of the palm. The *cubit* is thus, for the mensuration of matter, naturally divided into 24 equal parts, with subdivisions of which 2, 3, and 4 are the factors; while, for the mensuration of *distance,* the *foot*

will be found at once equal to one-fifth of the pace, and one-sixth of the fathom.[6]

In other words, with the digit (or the fingerbreadth) as the smallest unit of this natural system, we obtain the following table[7] showing how these units were related to each other:

Digit	
Palm (or handbreadth)	4 digits
Span	12 "
Foot	16 "
Cubit	24 "
Step or single pace	40 "
Double pace	80 "
Fathom	96 "

It is interesting to observe that the term "cubit" is derived from the Latin word *cubitum*, elbow. Again, the Latin designation of the forearm was *ulna*, whence the English *ell* and the French *aune*. The history of the various ells, cubits, feet, and their subdivisions, used in the course of time, would constitute a long and imposing document.[8]

The Anglo-Saxon *faethm* means "embrace." The "mile" comes from *mille passuum*, a thousand paces, for the pace was a double step, and hence a little over five English feet. The "fingerbreadth," used by both Greeks and Romans, caused the use of the word *digit* (Latin, *digitus*). The "palm" is based on the Latin *palmus*. This unit has survived in our "hand," still used in measuring the height of a horse. The word "inch" is derived from the Latin *uncia*, the twelfth of a foot or the twelfth of a pound, which originally signified a small weight.

Naturally, the units derived from the body were supplemented by other convenient measuring devices. Thus, the "furlong" is supposed to come from the Anglo-Saxon *furlong*, meaning "furrow long." The word "yard" comes from the Anglo-Saxon *gyrd*, meaning a stick or a rod, whence also a "yardarm" on the ship's mast.[9]

[6] John Quincy Adams, *Report upon Weights and Measures*, published by Abraham Small, Philadelphia, 1821. Extensive quotations from this report will be found in Judd, C. H., *The Psychology of Social Institutions*, Chap. VII, "The Psychology of Precision," Macmillan Co., 1927.

[7] Hallock and Wade, p. 6. (The complete title is given in the bibliography at the end of this report.)

[8] See, for example, Nicholson, Edward, *Men and Measures*, Chap. II, IV, V, XVI, London, 1912.

[9] Concerning the history of the common linear measures, see Smith, D. E., *History of Mathematics*, Vol. II, pp. 640-642, Ginn and Co., 1925.

The Multiplicity of Standards. It is now easy to understand why so many different units of length were in actual use throughout the ages, and why they varied so extensively. Thus, the Attic foot was 12.137 inches; the Roman foot, 11.67 inches; the Rhineland foot, 12.356 inches; and the Amsterdam foot (used in colonial days), 11.146 inches. Throughout Europe there were dozens of kingdoms, principalities, and free cities, each with their separate systems. "Social conditions and traditions everywhere governed, and not only in different countries in the same region would there be different values for the same weights and measures, but also in different towns of the same state."

It has been estimated that at the close of the eighteenth century, in different parts of the world, the word "foot" was applied to 282 different units of length. It is not surprising, therefore, that a demand for uniformity should have asserted itself with ever increasing insistence.

The English Units of Length. The source from which the Anglo-Saxons derived their weights and measures is not particularly certain, yet they early endeavored to secure uniformity by enacting good laws, and in this they were so successful that they were enabled to maintain these weights and measures in their integrity despite the Norman conquest. In fact, they were specially recognized and preserved by a decree of William the Conqueror, which stated that "the measures and weights shall be true and stamped in all parts of the country, as had before been ordained by law." The standards of the Saxon kings which had been preserved at Winchester were, however, removed to London, where they were deposited in the crypt chapel of Edward the Confessor in Westminster Abbey, which later became known as the Pyx Chapel. With Winchester are associated the earliest Anglo-Saxon weights and measures, and their authority as standards is said to date back to King Edgar (reigned 958-975), who decreed that "the measures of Winchester shall be the standard." The unit of length was the *yard* or *gird*, which was identical with the *ell*, and as late as the reign of Richard II (1377-1399) the words *virga* or *verge* (yard) and *ulna* or *aulne* (ell) are found in the laws and official documents in Latin or Norman French, as the case may be, to denote the same unit of length. In addition to the purely Saxon measures there were those which had been brought by the Romans, and which, though incommensurable with Saxon measures, had survived and

become assimilated with the older measures. Among these were the *mile*, corresponding to the Roman *mille passuum*, the *inch* and the *foot*, which soon became recognized as purely English measures and to have their own fixed values.

In the Domesday Book (1086) we find the *Saxon yard* used as a unit of measure, and land thus measured is referred to as *terra virgata*, and shortly afterward, from the reign of Henry I (reigned 1100-1135), the tradition is current that the legal yard was established from the length of that monarch's arm. The most important early English legislation was contained in Magna Charta (1215), and laid stress on the principle of uniformity. This declaration of uniformity was considered so fundamental that it was subsequently repeated in numerous statutes in essentially its original form, and we find many acts passed as occasion demanded to carry out its manifest intention. This naturally involved the definition of the standards and measures, and from time to time statutes are found which supply us with more or less complete information about the measures of the period.

Unlike the measures of weight and capacity, there have been few changes in those of length from the times of the Saxons, and the earliest surviving standards of length, those of Henry VII (about 1490) and Elizabeth (about 1588), vary scarcely more than a hundredth of an inch from the present imperial yard. In fact, we find the Anglo-Saxon measures of length perpetuated on the same basis as is given in the statute of Edward II (1324), where there is a restatement in statutory form of what has since become the well known rule that three barleycorns, round and dry, make an inch, twelve inches a foot, three feet a yard (ulna), five and a half yards a perch, and forty perches in length and four in breadth an acre.[10]

The old Winchester Standards remained essentially unaltered from 1588 to 1824. In that year the new "imperial system" of weights and measures now generally in force throughout the British Empire was defined by a special Act of Parliament, the *yard* being made the official unit of length. The standards legalized in 1824 were injured at the burning of the Houses of Parliament in 1834. They were restored eventually in 1854. New legislation was added from time to time, especially the important Weights and Measures Act of 1878, which is now in force.[11] How the common American

[10] Selected from Hallock and Wade, pp. 30-37.

[11] See H. J. Chaney, *Our Weights and Measures*, Eyre and Spottiswoode, London, 1897.

units are related to the English, and how they were modified from time to time, is an interesting story which cannot even be sketched in these pages.[12]

The Struggle for a Universal and Permanent Standard. The idea of deriving an invariable unit from nature itself, one that could be determined with great accuracy and duplicated readily, was bound to come to the surface sooner or later. During the seventeenth century it finally assumed tangible form. Various methods of attacking the problem were offered, all of them based ultimately on the earth. The pioneer suggestion which finally led to the metric system is generally ascribed to Gabriel Mouton, Vicar of St. Paul's Church, Lyons. He was the first to propose, in 1670, a comprehensive decimal system having as a basis the length of an arc of one minute of a great circle of the earth. One of the units of this system was further defined by Mouton as corresponding to the length of a pendulum making 3,959.2 vibrations in a half hour at Lyons.

A long debate ensued as to the relative merits of the *arc* method and the *pendulum* method of deriving the fundamental standard. Thus, in France, Picard suggested (1671) the length of a pendulum beating seconds. Huygens (1673) also endorsed this unit. La Condamine (1747), realizing that the earth is not a perfect sphere and that as a result there would be variations in the length of a second's pendulum at different latitudes, proposed the use of a pendulum beating seconds at the equator.

This controversy engaged the ablest minds of the day in Europe and America.[13] The culmination was reached in 1790 when Talleyrand, then Bishop of Autun, induced the French National Assembly to take an active interest in the movement. A committee of brilliant men, consisting of Borda, Lagrange, Laplace, Monge, and Condorcet, was chosen to study the problem. This committee considered three possible units as the basis of a uniform and satisfactory system of weights and measures: the length of a second's pendulum, the quadrant of a great circle of the equator, and the quadrant of a great circle of the meridian. The committee selected an arc of a meridian rather than one of the equator. To quote:

[12] See John Quincy Adams, *Report upon Weights and Measures;* also the excellent treatment in Hallock and Wade, Chap. IV.

[13] The great service rendered to the United States by Thomas Jefferson, through his work on behalf of a decimal system of coinage, and through his enthusiastic support of the metric movement, should be stressed more definitely in the classroom. See *The Works of Thomas Jefferson,* Vol. VII, pp. 472-495, New York, 1884.

After an arc had been measured, the length of a *quadrant* could then be computed, and one ten-millionth of its length could be taken as the base or fundamental unit of length. The plan proposed by the committee was to measure an arc of meridian between Dunkirk, on the northern coast of France, and Barcelona on the Mediterranean Sea, largely because these two places were each situated at the sea-level in the same meridian, because they afforded a suitable intervening distance of about 9°30′, the greatest in Europe available for a meridian measurement, because the country so traversed had in part been surveyed trigonometrically previously by Lacaille and Cassini in 1739-1740, and furthermore because such an arc extended on both sides of latitude 45°. The committee outlined six distinct operations essential for the work. They were as follows:

1. The determination of the difference in latitude between Dunkirk and Barcelona.
2. The measurement of the old bases.
3. The verification and measurement of the series of triangles used in a previous survey, and extending the same to Barcelona.
4. The observation of the pendulum at 45° latitude.
5. Verification of the weight in vacuum of a given volume of distilled water at the temperature of melting ice.
6. Comparison of the old and new measures, and the construction of scales and tables of equalization.

The National Assembly adopted the recommendation of the Committee. The actual work was commenced in 1792 by Méchain and Delambre, through whose heroic devotion the task was finally completed. Such was the struggle which gave rise to the metric system.

The Metric System. The story of the metric system has been written so often and is so readily accessible that a few references must suffice at this point.[14] It would be a great help if the memorable report of Méchain and Delambre,[15] setting forth the early work necessary to establish the metric system, were made available in an English translation.

The agitation for and against the metric system is still going on in England and America. In this age of science, of the radio, and of countless electric appliances based exclusively on metric units, the usual arguments against the early introduction of the metric

[14] A brief statement of the essential facts will be found in Smith, D. E., *History of Mathematics,* Vol. II, pp. 648-650. For an extensive treatment see, especially, Hallock and Wade, Chap. II-X.

[15] Méchain and Delambre, *Base du Système Métrique,* 3 Vols., Paris, 1806-1810. A condensed German translation, by W. Block, appeared in the collection *Ostwald's Klassiker der Exakten Wissenschaften,* No. 181, Leipzig, 1911. Copious extracts from this translation can be found in W. Dieck's *Mathematisches Lesebuch,* 4. Band, Sterkrade 1, 1920.

system seem incredibly shortsighted. The interests of the metric system in America are now sponsored energetically by the Metric Association,[16] which has issued numerous explanatory pamphlets for general distribution and is carrying on an extensive publicity campaign.

Teachers should become more familiar not only with the history of the metric system, but also with its obvious and compelling advantages. The development of the metric system, as the culmination of an age-long struggle for uniformity and economy, has been called the greatest human achievement since the invention of printing. Its progress in the brief space of a century has been astounding. We are thus approaching the ultimate goal of giving to humanity the same common background for the art of measurement which, fortunately, we have long enjoyed in the domain of number. We ought to teach our students how to use it intelligently.

The Search for an Absolute Standard. Great as had been the precautions during the original surveying operations conducted to determine the "meter," subsequent researches showed that a slight error had been made in finding the latitude of Barcelona. As a result, the standard was in error by about 0.1 millimeter, a little more than a hair's breadth. The average length of the earth's quadrant, according to the best modern estimates, is about 10,002,100 meters. Hence the "meter" now in use does not conform strictly to the original definition.

Above all, the problem remained of securing an *invariable* unit of reference, since the permanence of the prototypes kept in Paris could not be guaranteed. Even this apparently insurmountable difficulty was finally overcome through an ingenious suggestion of J. Clerk Maxwell. As an absolute unit of length, he proposed the wave-length of some determined kind of light. The idea was actually carried out by Professor A. A. Michelson.[17] "Using three different kinds of light, namely, the red, green, and blue of the cadmium spectrum, he determined the wave-length of each, or the number of times this wave-length was contained in the standard meter." He found that, in terms of red radiations,

$$1 \text{ meter} = 1{,}553{,}163.5 \text{ wave-lengths.}$$

[16] The publications of the Metric Association can be obtained at very slight expense. Its offices are at present located at 156 Fifth Ave., New York City.

[17] See Chaney, H. J., *op. cit.*, p. 23; Stevens, J. S., *Theory of Measurements*, p. 1, New York, 1916; Hallock and Wade, pp. 260-266.

"The accuracy of this work is incredible, as the variation in the measurements was only about one part in ten million. Here then is an absolute measurement, which gives the length of a standard in terms of a natural unit, under conditions reproducible at any time." And thus a perfect cosmic standard has finally replaced the crude units of earlier days.

Part Two

The Direct Measurement of Line-Segments (Classroom Procedure)

The Need of Early Geometric Training. In the junior high school, intuitive geometry is now usually begun in the seventh grade and is continued in one form or another throughout the course. There is no reason, however, why geometric training should not be introduced much earlier, as is the case in nearly all European school systems. Fortunately, there is an increasing tendency in the best elementary schools to stress practical work in measurement as early as the fourth grade in connection with the manual training projects of the children. It is also gratifying that in the most progressive junior high schools the shop work and the courses in industrial or household arts offer splendid opportunity for continuous application and correlation. Before beginning actual classroom work in the field of linear measurement, the teacher should create the necessary atmosphere of anticipation or appreciation. This can be done by taking a few minutes each day for a rehearsal of the significance of measurement in everyday life, and for presenting, as dramatically as possible, the main facts suggested by the story of measurement. If these brief discussions are then followed by a socialized recitation based on these facts, the apathy often displayed by young pupils toward accuracy in all its forms will be eliminated to a very large extent, if not entirely.

Throughout this preparatory period the teacher should stress, with increasing emphasis, the unique importance of *linear* measurement. It is easy to show, in the first place, that in ordinary construction work, such as the making of furniture, boxes, and the like, in the manufacturing of clothing, shoes, automobiles, machines, we are nearly always concerned with the measurement of dimensions, that is, of line-segments. In other words, whether we wish to determine perimeters, or areas, or volumes, we are always funda-

mentally dependent on linear measurement. The pupil will then readily see that linear measurement is our great tool for exact and economic production in the industrial world. Second, we must determine distances and heights by linear measurement. That is, transportation, rapid transit, commerce, surveying, navigation, and all similar enterprises, presuppose the constant, skillful use of linear measurement.

A Desirable Technique. Turning now to the discussion of a desirable classroom technique, the writer wishes to state that his suggestions are intended for seventh grade pupils who have had little or no preliminary training in measurement. In the case of younger or older pupils any experienced teacher can readily make the necessary modifications.

It may not be superfluous to enumerate a few of the essential characteristics of an efficient classroom method. It should certainly be simple and practical, and it should not demand an elaborate equipment. It must be adaptable to large or small classes, and it must admit of reliable testing and checking.

After much experimentation, it has been found that the best results seem to be obtained if the work in direct measurement is carried on in four distinct, successive steps, as follows:

1. There should be a careful study of the measuring instrument.
2. There must be adequate practice in "prescribed" measurement.
3. Definite provision must be made for the development of accuracy, through "controlled" measurement.
4. The skill thus acquired should then be exercised extensively through suitable applications.

These four steps will now be considered in detail.

I. The Study of the Instrument

The Ordinary Ruler. The simplest and the most familiar measuring instrument is the ordinary ruler, and the numerous linear measurements which occur in everyday life render its constant use indispensable. For that reason, presumably, the mistake is often made of assuming that every seventh grade pupil has had a sufficient number of contacts with measurements of a practical sort to enable him to proceed at once to measurement "projects." It is true that the average home usually boasts of a yardstick, that many boys

have casually observed construction work in the street, or in shops and factories, and that, similarly, many girls have watched their mothers as they used tape measures in dressmaking and sewing. While this apperceptive background is very valuable, it is not a sufficient reason for expecting immediately an efficient use of the ruler. Besides, there are children in every classroom who do not possess even this rudimentary first-hand acquaintance with weights and measures. In spite of this, textbooks often create the impression that the real point of departure should be a suitable measurement "project." How interesting it must be to pupils to be initiated into *real* measurement at once by "laying out a tennis court," or "surveying a playground," or "planning a house"! It is so completely in harmony with the popular tendency to "do things" and "not to worry about petty details," that we allow children, in the early grades, to attempt things which are never mastered and which, as a result, are subsequently a continuous source of trouble.

Standard tests, however, tell a very plain story. They show that the fundamental skills are acquired slowly, and that the cardinal aim of teaching, in this field, should be the prevention of errors or wrong ideas at the beginning.[18] If measurement is to be regarded as a skill, it will require a technique which carries the pupils beyond mere "exposure," and beyond a few uncontrolled "diffuse movements."

The point of view represented in these pages is that measurement, "the master art," ranks with computation as one of the basic "adaptations" that cannot safely be neglected in the training of any child. If this is true, it is necessary to "make haste slowly," especially in large classes, and to reject the "project method" as the exclusive means of developing skills of any type.[19] It is clear, however, that supplementary projects should certainly be introduced in due time for purposes of motivation and genuine application.

The first step, then, in the effective teaching of linear measurement is the study of the appropriate measuring instrument which, in this case, is the ruler. As to equipment, it is assumed that each

[18] Myers, G. C., *The Prevention and Correction of Errors in Arithmetic*, The Plymouth Press, Chicago, 1925; Buckingham, B. R., *Research for Teachers*, Chap. VIII, Silver, Burdett and Co., 1926.

[19] See the confirming statements of Superintendent C. Washburne of Winnetka, Ill., in the *Twenty-sixth Yearbook of the National Society for the Study of Education*, Part I, Chap. XI, especially p. 227.

pupil can be provided with a ruler showing the common English and metric units. On the ordinary inch scale, the inches are divided into halves, fourths, eighths, sixteenths, and sometimes into thirty-seconds. The metric scale shows centimeters, half-centimeters, and millimeters. The following combination scale has been found particularly serviceable, although a simpler form may be used.

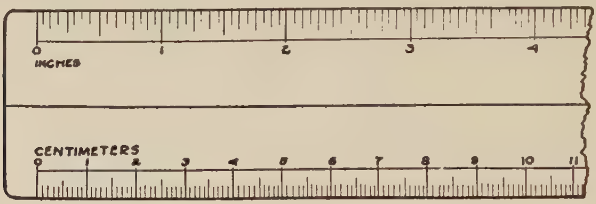


FIG. 1 (*reduced* ⅓)
In this diagram, the centimeter scale was reversed, for convenience of reference.

In every classroom, there should be, of course, at least one yardstick and one meter stick. A combination instrument is often used by science classes. A tape is also very desirable. In addition, the teacher, or a group of pupils, should prepare a strip of paper about one inch in width and at least eight feet long, which is graduated into feet and inches. This strip should be displayed prominently before the class on the upper edge of the blackboard frame, and should remain in position for several weeks. A similar strip should be used eventually for the metric units. If possible, each pupil should make such a strip for personal use. The making of such instruments is in itself valuable work.

For two or three days the teacher should then provide exercises such as those outlined below. A few minutes of energetic, enthusiastic work of this sort, each day, will mean more than a single, extensive recitation of the usual type. In the interest of clearness, the specific objectives to be kept in mind in these exercises are definitely enumerated.

Getting Acquainted with the Units. Let individual pupils come forward and point out on the yardstick, or on the paper strip, specified lengths, emphasizing first the larger units, and then the smaller units.

Ex. 1. Show a length of 1 ft.; of 2 ft.; of 6 ft.
Ex. 2. Show a length of 1 yd.; of 2 yd.

Ex. 3. Lay off, on the frame of the blackboard, or on the floor, a distance of 5 ft.; of 10 ft.

Ex. 4. Mark off, on the wall, a distance of 2 yd.; of 4 yd.

(Let each pupil hold up his ruler and then point out specified distances involving inches.)

Ex. 5. Point out a distance of 3 in.; of 7 in.; of 9 in.

Ex. 6. On the yardstick, you may show a distance of 12 in.; of 18 in.; of 24 in.; of 30 in.

Ex. 7. On the paper strip attached to the blackboard you may point out a distance of 3½ ft.; of 4½ ft.; of 50 in.; of 63 in.

(The subdivisions of the inch should then be examined, particular attention being directed to the fact that these divisions represent successive bisections. A preliminary reference to decimal divisions of the inch may also be introduced.)

Ex. 8. Lay your pencil on your ruler and show on the pencil a distance of 3½ in.

Ex. 9. (Comparison) Place the edge of a sheet of paper on your ruler, and mark off on it a distance of 4½ in. Then tear off a strip of paper of that length. All these strips should then be collected. It will be easy to find out which of these strips are of the specified length.

Ex. 10. On the ruler point out a distance of 5¼ inches. Then proceed as in Ex. 9.

These exercises should be continued until the pupils are reasonably familiar with the customary units. It must be left to the teacher's judgment whether metric units should receive similar attention at once or at a later date.[20] Both plans have been tried with success.

Estimating Lengths. The primary purpose of "estimating" at this point is the cultivation of a desire for more exact measurement. One thing seems fairly certain, namely, that if the present generation of pupils is taught to use the metric system intelligently, its final adoption in this country ought to be assured.

Ex. 1. (The pupils are told to hold their rulers horizontally in such a way that they are looking at the reverse side of the instrument. See Fig. 2.)

Try to point off on the (ungraduated) side of the ruler a distance of 3 in. Fold a narrow strip of paper and partly wrap it around the ruler in such a way that the paper can be made to slide easily back and forth. Move this paper indicator to the point which, in your opinion, marks off a distance of 3 in. Holding the paper firmly in its position, turn the ruler over and see

[20] This is also true of the decimal divisions of the foot and the inch. Rulers graduated into tenths of an inch are not used commonly in everyday life. A good substitute can readily be made by cutting off a strip of squared paper showing decimal divisions and pasting it on a piece of cardboard or on the back of an ordinary ruler. A combination instrument showing decimal divisions of the inch was developed by Miss H. S. Poole of the Lafayette High School, Buffalo, N. Y., and is successfully used in that school.

how nearly your *estimated* length of 3 in. agrees with the actual distance marked off.

Ex. 2. Repeat Ex. 1, trying to mark off a distance of 4½ in. Check with the paper indicator.

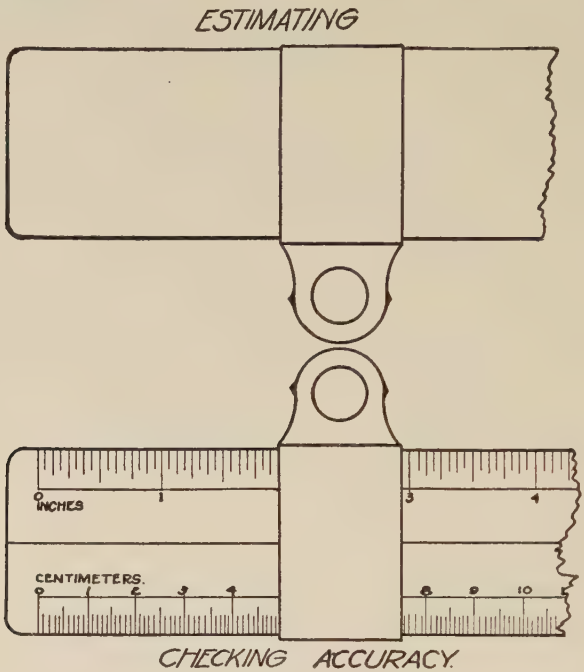


Fig. 2 (*reduced* ⅓)

Ex. 3. By means of a table such as the following record your success in making the readings suggested by the teacher.

Estimated Length	Actual Length	Actual Error
3 in.	3½ in.	½ in.
4 in.	......	.....
5½ in.	......	.....
6 in.	......	.....
9 in.	......	.....
2 in.	......	.....

Ex. 4. Estimate the dimensions of your book to the nearest inch. Check by actual measurement.

Ex. 5. Estimate the length of your pencil to the nearest inch. Check.

Ex. 6. Estimate the dimensions of your desk to the nearest inch. Check.

If time permits, "estimating" may be extended to other familiar objects in the room, such as pictures, bookcases, and windows. It

will be found that skill in estimating is of slow growth. Moreover, the position of the object will affect the results. Horizontal distances are, for obvious reasons, estimated more easily than vertical distances. Skill in estimating small segments will not automatically result in the development of similar skill in dealing with long segments. Hence each position and each typical length must be dealt with separately.

Copying the Scale Divisions. By this time the pupils have acquired sufficient familiarity with the units and with the technique of measurement to begin systematic work in their individual notebooks. The usual precautions and admonitions as to sharp pencils, correct position, and cleanliness are in order at this point.

Ex. 1. On a straight line mark off, in succession, 5 inch divisions. Write opposite each inch division the corresponding number.

Ex. 2. In the first inch division, mark off half inches; in the second division, mark off fourths; in the third, mark off eighths, and so on.

Ex. 3. Check the accuracy of your work by sliding the ruler along the line to see if the divisions are correctly marked when you use a different starting point on the ruler.

A similar plan may be used, either at this time or at a later date, in dealing with decimal divisions of the inch, or with the metric units.

II. Prescribed Measurement

Testing Acquired Skills. The study of the ruler, and especially the exercises in estimating, will have stimulated in the pupil a desire to try out his newly acquired skill. Even now, however, it is not advisable to rush too quickly into indiscriminate measurement activities. It is much better to direct and supervise the pupil's further progress in this field for at least two or three weeks, before encouraging a more spontaneous or independent approach. Again, a short drill period of fifteen minutes each day is preferable to lengthy discussions.

The following simple types of directed or prescribed measurement exercises will be found convenient for this purpose.

Crude Measurements. A paper strip showing inch divisions should be attached to the upper frame of the blackboard, as a "line of reference."

Ex. 1. On the blackboard draw horizontal lines of the following lengths: 3 ft.; 31 in.; 4 ft.; 29 in.; and so on.

Ex. 2. On the blackboard draw vertical lines of the following lengths: 2 ft.; 17 in.; 27 in.; 2½ ft.; and so on.

Ex. 3. Draw oblique lines of given lengths.

Ex. 4. Draw horizontal lines of various lengths on the blackboard. Estimate the length of each, to the nearest inch. Check by measurement in each case and state the actual error.

Ex. 5. Draw vertical lines of various lengths and proceed as in Ex. 4.

Ex. 6. Draw oblique lines of various lengths and proceed as in Ex. 4.

More Refined Measurements. In their notebooks the pupils may then work out exercises which deal with more refined measurements such as the following:

Ex. 1. Draw four horizontal lines. On these lines mark off, respectively, the segments a, b, c, d, of the following lengths:

$$a = 2 \text{ in.},\ b = 3\tfrac{1}{4} \text{ in.},\ c = 2\tfrac{7}{8} \text{ in., and } d = 3\tfrac{5}{16} \text{ in.}$$

(If squared paper showing tenths of an inch is available, an admirable preparation for graph work is possible at this point.)

Ex. 2. On a sheet of squared paper, mark off segments of the following lengths:

(1) 2.5 in. (3) 4.1 in.
(2) 3.2 in. (4) 1.8 in.

Ex. 3. Measure the length and the width of a specified sheet of paper

(1) to the nearest inch. (3) to the nearest ¼ inch.
(2) to the nearest ½ inch. (4) to the nearest ⅛ inch.

Ex. 4. Measure the length and the width of the blackboard to the nearest foot; to the nearest inch.

Scale Drawing. Ordinary scale drawings afford excellent drill in directed measurement. In such a drawing, a distance of a foot is represented by an inch or by a specified fraction of an inch. In other words, the actual dimensions must first be translated into the scale values, and the reduced (or enlarged) segments which result must then be drawn.

Ex. 1. In the following table, using the given scale, fill in the missing numbers.

Scale: 1″ to 1′.

Actual length	1′	4′	10′	15′		
Scale length	1″	4″	10″		17″	20″

Ex. 2. A rug is 4 ft. by 3 ft. Using a scale of 1 in. to 1 ft., what are the dimensions of the rug in the scale drawing? Draw the resulting figure.

Ex. 3. Scale: ½″ to 1′.

Actual length	2′	6′	10′	15′		
Scale length	1″	3″			7″	10″

Ex. 4. Give similar exercises involving other scales.

Ex. 5. A group of seventh grade pupils collected plans and blue prints of houses found in newspapers and periodicals. They compared the dimensions of the living rooms in these plans. They made a table of the dimensions found and then drew plans of the rooms which they considered large enough for a family of four persons. Here is a portion of the table. Select any one of the rooms and draw its plan, using the given scale.

Living Room	Scale
1) 12′ x 12′	¼″ = 1′
2) 12′ x 14′	¼″ = 1′
3) 14′ x 10′	½″ = 1′
4) 11′ x 11′	½″ = 1′
5) 22′ x 14′	⅛″ = 1′

Ex. 6. A rectangular rug is 48 in. wide and 56 in. long. Make a drawing representing it. Use a scale of 1 to 10.

Bar Graphs and Line Graphs. Obviously, whenever we represent statistical data of any sort by line-segments, we have an application of prescribed measurement. In each case we must first express the given data in terms of the scale unit chosen, and must then lay off segments picturing the resulting values.

As soon as the original table of values involves large numbers the choice of a suitable scale unit is imperative. Teachers and textbook writers often make the serious mistake of presenting tables which are not graded as to difficulty. Nor do they offer a reliable method of dealing with large numbers. It is here that the training discussed in these pages can be developed and utilized. We should do all that we can to encourage improvement among our teachers and writers in regard to such work.

Naturally, the simplest and most effective method of handling large numbers graphically is based on the use of decimal scales. The first exercises should be limited to numbers lying within the range 1 to 100. Then we may pass to numbers between 100 and 1000.

Finally, then, values larger than 1000 may be considered. As soon as the numbers to be represented graphically become large, the vexing problem of "approximating" or "rounding off" arises.

Psychologically, it is best to let the first graphs "talk distances" in order that the pupil may remain in a familiar field. That is, the first graphic problems should picture lengths or heights.

The method to be used in extending prescribed measurement to the case of graphs may be inferred from the following exercises:

Ex. 1. On five successive days Mr. Smith recorded the following mileage made by his automobile: 84 mi., 96 mi., 45 mi., 63 mi., 99 mi. Represent these distances by line-segments.

Using a scale of 1 in. to 10 mi., we obtain segments of the following lengths: 8.4 in., 9.6 in., 4.5 in., 6.3 in., 9.9 in. These distances can be easily pictured on decimally divided squared paper.

Ex. 2. The following table gives the distances from New York of radio broadcasting stations in the cities mentioned:

City	Miles from New York	Segment
Toronto	335	....
Pittsburgh	330	....
Los Angeles	2400	....
Portland, Ore.	2435	....
Denver	1615	....
San Francisco	2550	....
Washington	210	....
Chicago	705	....
Boston	205	....
Cleveland	400	....

Using a scale of 1 in. to 100 mi., how long a segment would be necessary in each case to picture the given distance?

Solution. Here, the segments to be pictured have the following lengths:

3.35 in., 3.30 in., 24 in., 24.35 in., 16.15 in., and so on.

These values compel us to consider a method of "rounding off."

Method of "Approximating" or "Rounding off." This subject, when presented carefully, is readily comprehended by seventh grade pupils. Thus, if the scale is 1 inch to 100 miles, a distance of 784 miles is represented by a segment of 7.84 inches. Since it is

$$\begin{array}{lll} \quad 7.90 & & \\ \rightarrow 7.84 & & \\ \quad 7.80 & \rightarrow 7.80 & \rightarrow 7.8 \end{array}$$

difficult to locate hundredths of an inch correctly in a drawing, we may "round off" the result, or "approximate" it, as shown above.

This means that 7.84 lies *between* 7.80 and 7.90. It is nearer to 7.80 than to 7.90. Hence, if we desire accuracy to the nearest tenth only, we may use 7.8 instead of 7.84.

If the required segment has a value ending in 5, such as 6.25, the rule generally followed is *to approximate in such a way that the last figure is even.* Thus, 6.25 becomes 6.2, while 6.35 becomes 6.4, and 6.55 becomes 6.6.

Hence, a distance of 335 miles, in the above table, may be represented by a segment of 3.4 inches, while a distance of 705 miles is represented by 7.0 inches.

III. Controlled Measurement (Accuracy)

Extent of the Work. For retarded or "slow" pupils, the work outlined thus far represents the maximum safe requirement in the seventh grade. Normal or accelerated pupils can and should go considerably beyond this goal. With the aid of a few sheets of squared paper graduated to tenths of an inch, or to millimeters, it is not difficult to improve their accuracy record. Errors can easily be reduced to an amount not exceeding a sixteenth (or 1/20) of an inch on the inch scale, and to a millimeter on the metric scale. An even greater degree of accuracy can be obtained within a week, at the rate of ten or fifteen minutes each day. It is only necessary to devise a method of checking or controlling each measurement. Within the range specified, the procedure which we shall explain in the next few pages has been found from classroom experience to be both simple and efficient.

Motivation. Our entire industrial era is really based on precision measurements. Quantity production, in many key industries, would become impossible without a high degree of accuracy. Many conveniences, formerly considered luxuries, are now within the reach of all, simply because they can be put on the market quickly and economically by means of standardized and controlled methods of production. This is true, for example, of automobiles, vacuum cleaners, victrolas, radio sets, watches, fountain pens, and innumerable other miracles of quantity production. It is these conveniences and improvements which differentiate the modern world so completely from the period of a hundred years ago. Hence it is essential, whether for cultural or for vocational reasons, to become acquainted with some of the devices which have made these modern miracles possible.

Precision Instruments. The automobile, long familiar to every boy and girl, is an almost classical example of precision engineering. A vast number of parts must be manufactured and assembled before the finished car can be sent out from the factory. An army of workmen housed in different buildings, and stationed at an almost endless array of machines, assists in turning out these thousands of parts. Unless this necessary division of labor were accompanied by constant and reliable checking at every point, the final outcome would be disastrous. Constant provision is made against errors in measurement by the use of an interchangeable precision apparatus which enables two or more mechanics to make the same measurements with an error of not more than a thousandth or a ten-thousandth of an inch.

A visit to a machine shop or a factory will serve far better than mere words to impress upon the pupils the importance of extreme accuracy in many lines of work. Specific illustrations of the degrees of accuracy demanded and obtained in various industries are always appreciated. Thus, the steel balls for the bearings of the L. C. Smith and Brothers typebars must not be less than 622 ten-thousandths nor more than 628 ten-thousandths of an inch in diameter. Between these possible limits there are seven distinct sizes, and fifteen balls of any one of these sizes may be used in the construction of a perfect typebar bearing. The seven sizes of balls used differ from one another by one ten-thousandth of an inch, but they are never allowed to be mixed.

If possible, the teacher should secure an actual micrometer caliper for demonstration purposes. It is easy to obtain a micrometer graduated either to thousandths (or ten-thousandths) of an inch, or to hundredths of a millimeter. Directions for using such instruments may be found in almost any shop manual. (See Fig. 3.)

A classroom demonstration which never fails to arouse interest consists in measuring the thickness of a sheet of paper with a micrometer. When it is seen that the thickness varies between 0.002 in. and 0.004 in., the pupils begin to understand for the first time what an accuracy of 0.001 in. really means. For similar reasons, the question is then raised as to whether it is possible to measure the thickness of a human hair, reference being made to the common phrase, "a hair's breadth." The breathless and almost frantic eagerness with which otherwise phlegmatic boys and girls suddenly pull out and offer for experimentation samples of their

own hair leaves no room for doubt that the subject has been motivated in a very real and unexpected sense. The samples submitted

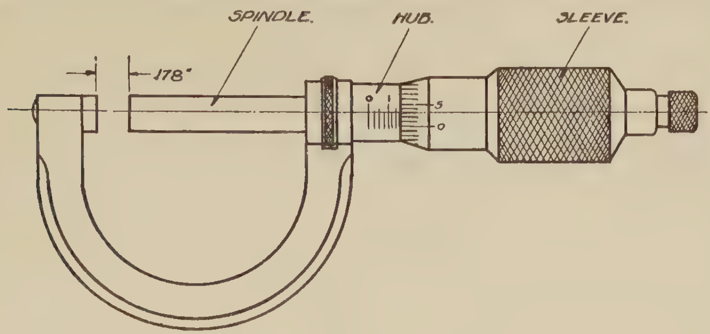


FIG. 3

In the accompanying diagram of a micrometer a reading of 0.178 in. is shown. The graduations on the hub conform to the customary pitch of the screw, namely 40 to the inch. Hence each division on the hub is $\frac{1}{40}$ in., or 0.025 in. Every tenth of an inch is numbered 0, 1, 2, etc. The beveled edge of the sleeve is graduated into 25 parts. When 25 of these graduations have passed the horizontal line on the hub, the spindle has made one revolution and hence has moved 0.025 in. Accordingly, when the spindle moves only far enough to cause one graduation mark of the sleeve to pass the horizontal line of the hub, it will have moved $\frac{1}{25}$ of 0.025 in., or 0.001 in. The distance between the graduations on the sleeve is great enough to permit half and quarter thousandths of an inch to be readily estimated.

usually reveal a diameter fluctuating between 0.001 in. and 0.002 in. In speaking of "a hair's breadth," therefore, popular diction long ago furnished a very acceptable standard of accuracy.

Practice in Estimating Tenths with an "Indicator." The simple device in Fig. 4 will quickly develop skill in dealing with

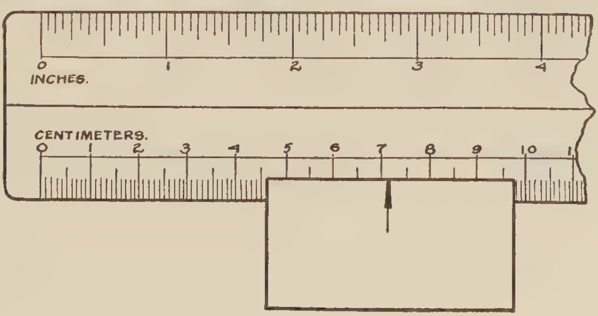


FIG. 4 (*reduced* $\frac{1}{3}$)

tenths of an inch or tenths of a centimeter, and will prepare the way

for dealing effectively with hundredths, provided a sufficient amount of practice is given the students in this work.

On a card or strip of paper draw a pointed line or arrow at right angles to the edge. Place this edge on a decimalized scale in such a way that the smallest graduations are hidden, but not the marks indicating inches and half-inches.

The pupil *estimates* to the nearest tenth the exact position on the scale which is pointed out by the arrow. Then, carefully sliding the card until the smallest divisions on the scale are exposed, it is possible to check each estimate at once. After making ten or more such controlled readings, the pupil will begin to appreciate the fact that measurements are only approximate and that various sources of error may affect the reading.

Practice in Estimating Tenths and Hundredths, Based on Squared Paper. The use of an indicator is helpful, but not essential. Equally good results can be obtained by depending entirely on squared paper. For the sake of greater clearness, the discussion will be limited to the inch scale. Each pupil should be provided with inch paper graduated into tenths of an inch.

In order to focus the attention of the entire class on the process to be studied, it is well to use greatly enlarged blackboard diagrams. No other device has been found as effective. It is then a very simple matter to illustrate and emphasize the various possible cases which may occur in the measurement of segments, as suggested in the following paragraphs and by Fig. 5.

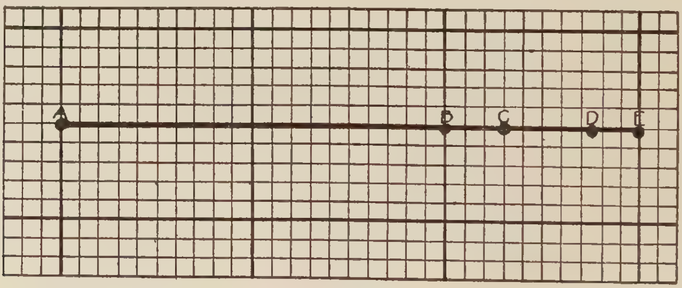


Fig. 5

Case I. What is the length of *AB*? The answer, evidently, is two inches.

Case II. How long is *AC*?

Here, the point *C* is at the third division of the inch extending

from B to E. Each of these small divisions is one tenth of an inch. Hence the length of AC is 2.3 in.

Case III. How long is AD?

The point D falls *between* the seventh and the eighth division of BE. Hence the length of AD lies *between* 2.7 in. and 2.8 in. In this case one of three plans may be followed:

1. We may use a more accurate instrument to measure AD.
2. We may select either 2.7 in. or 2.8 in. as the length of AD, according as D lies nearer the seventh or the eighth division mark of BE. It is clear, however, that a slight error is made in doing so. To indicate this, it is customary to say that the length of AD is about 2.7 in., or approximately 2.7 in.
3. Finally, we may imagine each of the ten small divisions of BE divided again into ten equal parts, thus dividing BE into 100 equal parts. To understand clearly the next step in measuring AD, suppose that the whole figure could be enlarged, or seen through a magnifying glass. The small division in which D is located would then appear as in Fig. 6. We can then estimate the

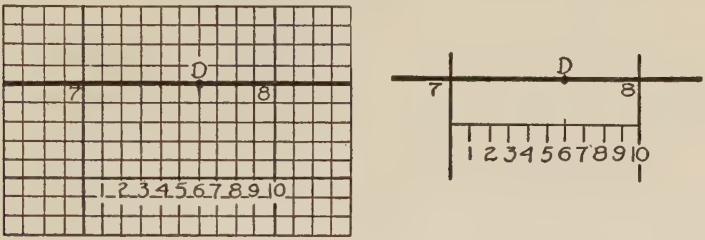


FIG. 6

position of D on this scale of imagined divisions. Thus, if D falls opposite the sixth of these new dividing marks, it is evident that AD extends 0.06 in. beyond the value of 2.7 in. Hence the length of AD is 2.76 in.

Since, however, we merely estimated the position of D, even the value 2.76 in., though more accurate than 2.7 in., is only approximately correct.

The process used in finding the length of AD is necessary in all practical measurements. A small error is usually made, whatever scale is used. Hence, *all measurements should be considered as only approximately correct.*

As soon as an appreciable number of pupils seem to comprehend the ideas outlined above, the teacher may begin to assign exercises such as the following.

Ex. 1. On inch paper draw the rectangle $ABCD$ as shown in Fig. 7. Measure the diagonal AC, in inches, to the nearest tenth; to the nearest hundredth.

Solution. With the aid of compasses, lay off AC along the base line, making $AE = AC$. Using the plan discussed above, we find that AE is 2.2 in., nearly; or 2.24 in., nearly.

Method of Checking. Now, it is easy to control or check this result by applying the Pythagorean Rule. For, in Fig. 7, *AC,* in inches, is readily seen to be $\sqrt{5}$. By referring to a table of square roots, the teacher can check every exercise of this type to

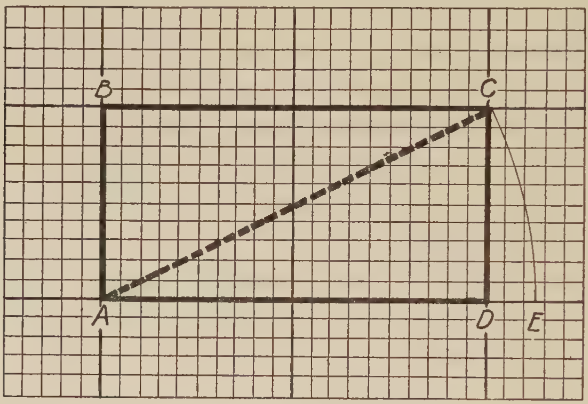


Fig. 7

any desired degree of accuracy. Since it is not possible, on squared paper, in the seventh grade, to get results beyond an accuracy of hundredths, the teacher must be prepared for errors which may be as large as 7 or 8 hundredths of an inch. Corresponding remarks apply in the case of metric units. A sufficient number of exercises should now be given to secure a fair degree of confidence and skill on the part of the pupils. This work should not be hurried. One or two exercises a day, for a week, represent a desirable rate of progress.

Ex. 1. Find the length of *AC,* in the rectangle *ABCD,* if its dimensions are: (1) 5 and 6; (2) 6 and 7; (3) 9 and 10; (4) 6 and 8; (5) 5 and 12; (6) 1 and 7.

Ex. 2. On squared paper draw a square *ABCD,* making its side 3 cm. long. How long is *AC*?

(*AC,* to the nearest tenth, is 4.2 cm.; to the nearest hundredth, it is 4.24 cm.)

Ex. 3. How long is the diagonal *AC* of the square *ABCD,* if *AB* is: (1) 5 cm.; (2) 6 cm.; (3) 10 cm.

Answers: (1) 7.07 cm.; (2) 8.48 cm.; (3) 14.14 cm.

IV. Measurement Projects. ("Free" Measurement)

Value of Projects. The project movement, in spite of all its obvious flaws, has at least served to call attention to the imperative need of linking the more or less artificial work of the classroom

more closely to actual life situations. To be sure, projects can never become the sole basis for the acquisition of skills. That is the delusion which has almost discredited the entire "project and activities" doctrine. Nevertheless, whenever a skillful and enthusiastic teacher succeeds in introducing such supplementary applications for the purpose of motivating, reviewing, or summarizing an ordinary unit of classroom work, the result usually is very encouraging.

Project-Problems. The following project-problems are submitted by the writer as illustrative samples of simple types of applications that may readily be attempted by seventh grade pupils at this point.

Prob. 1. The pupils in a health class had learned that in a good school the number of square feet of floor space for each pupil must be at least 15. They found that the standard classroom in their city was 30 ft. long and 23¾ ft. wide. How many pupils can safely be assigned to one class in such a room?

Prob. 2. Measure the floor space of your school room and see whether the size of your class meets the above standard.

Prob. 3. In a well lighted school, the per cent which the window area is of the floor area is at least 15. Thus, if the floor area is 700 sq. ft., the window area must be at least 105 sq. ft. Find out whether your classroom meets that standard.

Prob. 4. Each pupil of a science class measured the width of ten leaves of a certain tree. Then the average of these ten measurements was found. Try to repeat that experiment, selecting the leaves of any tree or shrub for that purpose.

A Seventh Grade Measurement Project. A more ambitious project, developed in the West Junior High School, Cleveland, Ohio, may be of interest. It was completed by a class of seventh grade B pupils, of the Y group. The following description is based on a report furnished to the writer by the teacher, Miss Hazel M. Miller.

This project I considered highly successful, primarily because it was suggested by the pupils. It was led up to as a result of my having read of the school board's appropriation of a considerable sum of money for the improvement of old buildings, and wondering about how many rooms could be improved with that amount. The question arose, "What would it cost to redecorate a room of this size?" And so the project was on. After an explanation of the situation, suggestions such as the following were offered by the pupils:

(1) We must measure the room.
(2) We must get prices of oil, paint, and labor.
(3) We must find out how long it would take.

The pupils were divided into six groups, each group being given a definite problem in measurement, such as the length, the width, and the height of the room, door measurements, blackboard space, window space, etc.

Then scale drawings were made of each wall, of the floor, and of the ceiling. The necessary areas (floor, windows, blackboard) were computed.

Boys reported on the cost of labor and of materials for oiling the floor. The whole class then estimated the cost. Reports on the price of paint, labor, etc., for refinishing walls and ceiling were made by the son of an interior decorator. Girls suggested sash curtains, secured prices, and estimated the cost. This work was all done neatly, and assembled in booklet form.

It took three weeks to complete the project, including all preparatory instruction necessary to secure intelligent coöperation. But the pupils learned to measure accurately and to compute areas, and at the same time had considerable practice in the fundamental operations. Moreover, they were responding to an actual life situation. Last, not least, they became more appreciative of the expense caused by their carelessness in dealing with school property.

A Correlated Sixth Grade Project. As was stated above, there is no reason why the manual training activities of the elementary school, as well as the lessons on weight and measures, given in every arithmetic class, should not be made productive of more significant mathematical results. A very successful correlating project was recently completed (December, 1927) in one of the elementary schools of Rochester, N. Y. The description which follows was furnished by Miss Florence Greenberg, the teacher in charge of the work in that school.[21]

BUILDING THE NEW SCHOOL

Submitted by Miss Florence Greenberg,
School No. 34—Miss E. H. MacLachlan, Principal

Our long-promised new school was at last being built. So much interest was shown in the new building going up right outside of the classroom window, and so much spare time was spent in watching the men work, that the sixth A class decided to make use of their observations and to build the school in miniature.

The contractor loaned the class a set of blue prints, and the work began. The scale of ¼ inch to one foot was decided upon, and was used throughout the entire project for every part. Then different committees were chosen to work on the separate parts.

For the framework we chose corrugated cardboard boxes. They were painted a brick color, and lines were drawn with white ink to represent bricks.

[21] As a member of a special Arithmetic Research Committee, organized in connection with the Rochester Coöperative School Survey, Miss Greenberg became interested in the possibilities of supplementary projects such as the one which is here described.

The parts were put together with adhesive tape. The floors, roof, and windows were made of isinglass, so that the inside partitions and rooms may be seen in the finished structure.

The primary object of the work was to supply the class with real problems in arithmetic. This was done in the following way. Each day, the children kept individual records of the work they accomplished and of the problems they encountered. Then, when the mechanical part of the work was finished, they decided to make a book which would include everything pertaining to their work. In this book they have their table of contents, a page of acknowledgments to the people outside the class who supplied them with material used, a list of all measurements made, a list of the cost of materials and of the men's wages, their original problems, the record of their work (or a "diary," as they called it), their own judgment of the project, and finally the index.

The cover and title page of the book were made during a regular art lesson. Letters of thanks were written during an English lesson. Hence, in spite of the fact that the school is run on a semi-departmental basis, the above subjects, along with penmanship, spelling, and reading, were closely correlated with the work. This, of course, can be done only if one has the full coöperation of the other teachers.

Although the sixth grade course of study does not include geometry, much incidental geometric training was secured. In the course of their work the children discovered and used the rectangle, the triangle, the circle, the semicircle, the cylinder, the rectangular solid, and the octagon (in the chimney). They noticed that the rectangle was used a far greater number of times than any other figure. They discovered that they could find the area of the roof, an irregular-shaped object, by dividing it into smaller rectangles, finding the area of each rectangle, and then getting the sum of the areas.

They got the idea of balance and symmetry, for in making the front of the building, the windows on one side exactly "matched" the windows on the other. The shrubbery was also placed "so that one side was the same as the other," again bringing out a symmetrical relationship. Illustrations of this kind would afford excellent motivation for the first lessons in intuitive geometry in the seventh grade of our junior high school course. As it is, they will well serve as a background for this work next term.

The scale used on most of the original blue prints was $\frac{1}{8}$ in. = 1 ft. This meant that the children had to change the given scale, $\frac{1}{8}$ in. = 1 ft., to $\frac{1}{4}$ in. = 1 ft., for their own working purposes. Later on, they had to change this scale back to the actual lengths appearing in the real building, for working their problems. The degree of accuracy with which they measured was within $\frac{1}{16}$ of an inch, and in some cases within $\frac{1}{32}$ of an inch. Needless to say, the pencil sharpener was overworked during this time, as the children found it necessary to keep their pencil points fine in order to get accurate measurements. The great test of accuracy came when the chairmen of the committees brought in their finished parts for assembling. There was much relief and joy when they found that the parts fitted almost perfectly.

The children spent two weeks of class time on the mechanical tasks. The rest of the work was done before and after school or in any spare time the

children had. However, we are still solving problems along with the regular work. It is over six weeks since the construction work started, and interest is still running high. All the work was done right in school, with the exception of the copper boilers which one boy's father made for him. The boy, however, got the measurements and reduced them to the right scale.

The children classified their problems and found that they came under the following headings: problems in area, in volume, in linear measure (denominate numbers), in finding costs of various items (dollars and cents). The processes included addition, subtraction, multiplication, and division of whole numbers, fractions and decimals. On looking over the course of study, I found that the work had correlated with it in every respect, and even included many topics not specially mentioned.

The cost of the project was five cents, spent for a package of pins used for ornamental balconies. The paint was willingly donated by a father who was a painter. Glue, paste, and plasticene were supplied by the school. The adhesive tape and isinglass were also donated by interested fathers. The boxes came from neighborhood grocery stores, thus making the actual sum of money spent only five cents.

There were twenty-eight children in this group. We could have kept twice as many busy, as each child had to work on two and in some cases three different committees. The ages of the children ranged from ten to fourteen years, the average age being between eleven and twelve years. Fifty-eight per cent of the class were of American parentage. The other forty-two per cent were German, Irish, Bulgarian, Canadian, Swedish, Scotch, and English. I do not think that nationality has a great deal to do with the success or failure of work of this type. Some of our projects are undertaken by classes having a high percentage of Italian children, and yet there is always a great deal of interest and concentrated work in connection with this sort of approach.

This brings me to my last point. The interest, zeal, and energy with which the class worked for the past six weeks have been remarkable. Some of the tasks required a great deal of patience, but all the work was done in such a fine spirit that what would ordinarily have been a burden was considered a pleasure. A vast amount was accomplished in a short time, and enthusiasm ran so high that children from other classes "dropped in" regularly and begged to be allowed to "do something." School life was indeed a joy to both the children and their teacher.

Part Three

Theoretical Considerations

Need of Technical Details. We shall now consider briefly some additional technical details which may be of value to classroom teachers for purposes of orientation. Elementary textbooks cannot be expected to offer extensive theoretical explanations. Unless the teacher can supplement these explanations, important ideas and

processes are likely to remain vague. Such terms as "degree of accuracy," "significant figures," "relative error," have a technical meaning which should always be explicitly stated. Needless to say, the uncertainty as to the precise significance of these scientific terms is due, at least in part, to the scarcity of elementary treatises on measurement suitable for schoolroom purposes.[22]

I. The Problem of Accuracy

Definition of Measurement. Direct measurement may be defined as the determination of the number of units of a certain kind required to equal a magnitude of the same kind. Hence measurement implies, first, the adoption of a unit of measure, and second, the operation of counting the number of times the unit is applied to the given magnitude.

Choice of Unit. It is obvious that in all practical problems the size of the unit should be chosen with reference to the size of the object to be measured. The dimensions of a building lot are ordinarily given in feet. Architectural data, as well as the dimensions of furniture, rugs, and other household equipment, are usually stated in terms of feet and inches. Larger distances are expressed in miles or kilometers. In astronomy, the "light year" is used very commonly, namely, the distance which light travels in a year at the rate of 186,000 miles a second. In machine work, dimensions are usually specified in terms of inches or centimeters. In science, the unit employed almost universally is the centimeter or its subdivisions.

Ideal Accuracy. The question arises whether the absolute accuracy of any measurement can ever be guaranteed. The answer, somewhat humiliating to human ingenuity, is "No." To quote:

> The average student is liable to have more or less difficulty in grasping the idea that accuracy is always a relative matter and absolute precision of measurement is an impossibility. Even the smoothest possible surface can be magnified enough so as to show that it contains irregularities everywhere.

[22] One of the best treatises is that of Tuttle, L., *The Theory of Measurements,* Philadelphia, 1916. Helpful discussions will be found in some recent college and high school textbooks. See, for example, Gale and Watkeys, *Elementary Functions,* pp. 72-78, Holt and Co.; Young and Morgan, *Elementary Mathematical Analysis,* Macmillan Co.; Karpinski, Benedict and Calhoun, *Unified Mathematics,* Heath and Co.; Swenson, John A., *High School Mathematics,* Macmillan Co.; Schultze and Breckenridge, *Elementary and Intermediate Algebra,* Macmillan Co.—The writer is indebted to these sources for some of the illustrative examples used in these pages and for valuable suggestions.

A geometrical plane certainly corresponds to nothing in reality, and perfect accuracy of number is just as much an imaginary concept. (Tuttle.)

Lord Kelvin has told us that one's knowledge of science begins when he can measure what he is speaking about and express it in numbers. Every year vast numbers of measurements are made in physical, chemical, and engineering laboratories, as well as in laboratories for advanced research. We are unable, however, to state concerning any one of these measurements that the result is absolutely correct. One of the most precise measurements in physical science is that of the wave-length of light. The wave-length of cadmium light, measured by a Michelson interferometer and a Rowland grating, was found to be

$\lambda c = 0.000064384722$ cm. (Michelson)
$\lambda c = 0.00006438680$ cm. (Rowland)

Or we may put it in another way,

1 meter contains 1553163.6 wave-lengths (Michelson)
1 meter contains 1553164.1 wave-lengths (Fabry and Perot)

These measurements were made by different observers using different methods. They are remarkable for their agreement and they give us the wave-length of light with sufficient accuracy for all purposes. But they are not correct, and it is not at all likely that we shall ever know the true length of a wave of cadmium light. (Stevenson.)

Estimating. A process of estimating arises in every measurement problem which involves large numbers or a high degree of accuracy. Whatever instrument is used, the observer must keep in mind, first, the degree of accuracy which the problem demands, second, the degree of accuracy which is possible with the given scale, and third, the extent to which the reading actually made conforms to either the required or the possible accuracy. A careful estimate will often save a great deal of trouble later.

Thus it is obviously unnecessary, and therefore not customary, to demand in dressmaking or in woodwork the precision that is essential in machine work or in any other form of precision engineering.

Certain and Uncertain Figures. Suppose that it is required to measure the dimensions of a large building lot and that a steel tape is used for the purpose. If the length is found to be 247 feet, correct to the nearest foot, one really means that the length is between 246 feet and 248 feet, but that it is nearer 247 feet. Here, the first two figures, 24, are *certain*, but the third figure is more or less uncertain because it represents an approximation of the actual length. Again, suppose that the length is determined more accurately, to the nearest tenth of a foot. If the length is found

to be 247.8 feet, the first three figures may be regarded as certain, while the 8 is uncertain.

In general, then, the last figure in a measured result, reading from left to right, is uncertain. It represents an *estimate,* and not an exact value.

Decimal Accuracy. If a segment is said to be 9.7 centimeters long, this means that it has been measured to *tenths* of a centimeter and that it lies between 9.6 centimeters and 9.8 centimeters, being nearer to 9.7 centimeters than to either 9.6 centimeters or 9.8 centimeters. More accurately, it lies between 9.6½ centimeters and 9.7½ centimeters. Similarly, if a length is written "9.70 cm.," it means that it has been measured to hundredths, and that it has been found to be between 9.69½ centimeters and 9.70½ centimeters, so that it can properly be "rounded off" to 9.70 centimeters. It is for this reason that scientists have adopted the agreement that 8 means "between 7½ and 8½," 8.0 means "between 7.95 and 8.05," and 8.00 means "between 7.995 and 8.005," and so on. In general, the rule followed is that *the statement of a measurement must be accurate as far as it goes, and that it should go far enough to express the desired degree of accuracy. Hence, no more figures should be written than are known to be correct, and no figures that are known to be correct should be omitted.*[23]

Significant Figures. "In arabic notation, the figures of which a number is composed, except for one or more consecutive ciphers placed at its beginning or end for the purpose of locating the decimal point, are called its *significant* figures."

In simple language, the significant figures of a measured result are those that *have* some meaning. This implies that, in some instances, a zero appearing in a measured result is significant, while in other instances it is *not* a significant figure. Suppose, for example, that the measured length of a segment is said to be 14 centimeters. If this result is then expressed in millimeters, it becomes 140 millimeters. As was stated above, 140 millimeters means "between 139.5 and 140.5," involving accuracy to the nearest millimeter; 14 centimeters means accuracy to the nearest centimeter. Hence zero, in 140 millimeters, was not obtained by measurement and is therefore not significant. If, on the other hand, the length of a segment is 30.4 centimeters, and is approximated to the nearest centimeter, namely, 30, the zero *is* significant, for it represents an

[23] Tuttle, *op. cit.,* p. 59.

observed value. In short, *a zero is significant if it represents an observed or calculated value; it is not significant if it does not represent an observed or calculated value, but merely serves to place the decimal point.*[24]

Thus, in such an expression as 0.0046, the zeros are not significant. Here the first significant figure is the first figure that is not zero, namely 4.

Approximating or "Rounding Off." Sometimes it is not necessary to retain all the significant figures appearing in a measured or calculated numerical result. In that case, the given value may be "rounded off" or approximated to the nearest tenth, or to the nearest hundredth, and the remaining figures may be dropped. The rule for dropping figures is that if the digit which is dropped is 6, 7, 8, or 9, the preceding digit is to be increased by one. Thus, 3.46 when rounded off to two significant figures, becomes 3.5; similarly, 4.92 becomes 4.9. The customary rule for rounding off *one-half* is to "record the nearest even number rather than the odd number which is equally near." Thus, 4.65 becomes 4.6; while 4.75 becomes 4.8. To quote:

> The reason for this procedure is that in a series of several measurements of the same quantity it will be as apt to make a record too large as it will to make one too small, and so in the average of several such values will cause but a slight error, if any. If the rule were that the half should be always increased to the next larger unit, the errors would not balance one another and the average would tend to be brought up to a larger value than it should have. The same advantage would of course be obtained if the nearest odd number were always used, but the even number has one slight additional merit, namely, that in case it should have to be divided by two, a recurrence of the same situation would be avoided. (Tuttle.)

As an application of the above rules, consider the number 7.346792. This number, correct to 4 significant figures, is 7.347; correct to 3 figures, it is 7.35; correct to 2 figures, it is 7.4. The nearest whole number is 7.

II. Degrees and Standards of Accuracy

Sources of Error. It is customary to distinguish between errors and mistakes, and to classify errors as either *constant* or *accidental.*

A "constant" error is one which affects all the measurements of a series in the same manner. It may be due to a faulty instru-

[24] Schultze and Breckenridge, *op. cit.*, p. 324.

ment, to a wrong use of the instrument, or to personal defects of the observer. "Accidental" errors are occasional small errors which tend to balance each other in a series of readings. "Mistakes," on the other hand, are errors which are caused by carelessness or other negative traits of the observer.

Possible Error. The figure which follows the last reliable figure in a measured result may be uncertain to the extent of several units. Thus, if a student estimates hundredths of a centimeter with the aid of a millimeter scale, his estimate will usually be in error by an amount varying between one and nine hundredths. That is, his actual error may be nearly one millimeter. A more careful observer, using the same scale, is not likely to make a mistake exceeding half a millimeter. In general, the *possible error* in a measurement, in the case of a trained observer, will usually not exceed *one-half of the smallest unit of the scale.* Hence, if the figures of a numerical value are significant and correct as far as they go, the given value cannot have an error larger than half of a single unit in the last decimal place, or five units in the place that would follow the last one that is written.

Thus, a length which is written 163.4 may be assumed to have an error of not more than 1 out of 1634; or, more accurately, an error not greater than ½ out of 1634. The importance of being a careful observer should be impressed on all students.

Relative Error. It is obvious that the actual or possible error in a measurement is of less importance than the *relative error.* The relative error is the ratio of the actual (or possible) error to the numerical value under consideration.

Thus, suppose that in measuring a line 20 inches long a pupil obtains as the result 19 inches. His actual error is 1 inch. The relative error is 1/20, or 0.05. If, on the other hand, in measuring a line 50 inches long the pupil also made an actual error of 1 inch, the relative error would be only 1/50, or 0.02.

Percentage Error. When the relative error in a measurement is expressed in per cent form, it is known as the *percentage error.* Thus, in the above examples, a relative error of 1/20 means a percentage error of 5%, while a relative error of 1/50 means a percentage error of 2%.

The relative error, and hence the percentage error, does not usually have to be computed with very great care. A fairly accurate estimate is sufficient. Such an estimate can often be obtained

by a simple mental calculation.[25] Thus, if in measuring a distance of 1 mile a surveyor made a possible error of 1 foot, the relative error would be 1/5280. This is very nearly the same as 1/5000, or 2/10,000, or 0.02/100, or 0.02 of 1%. The actual computation gives the value 0.00019, which is 0.019 of 1%.

"Significant Figures" versus "Decimal Places." It can easily be shown that it is the number of significant figures in a numerical result, rather than the number of decimal places, which determines the relative accuracy of that result. The table below shows this.

Given Value	Possible Error (assumed)	Percentage Error
87.4	0.1	1/874 = 0.1% (nearly)
8.74	0.01	1/874 = 0.1%
0.874	0.001	1/874 = 0.1%
0.0874	0.0001	1/874 = 0.1%

Degrees of Accuracy. We may assume that an experienced observer will be able to limit his possible error to one-half of the smallest unit of the scale. Now, if the measured result, expressed in terms of the smallest unit used, is denoted by x, the relative error is evidently expressed by the formula [26]

$$e = \frac{\frac{1}{2}}{x}, \text{ or } e = \frac{1}{2x}.$$

Using this formula, we may construct a table which shows how the degree of accuracy increases as the number of significant figures increases.

No. of Significant Figures	Range of Values of x	Range of Possible Percentage of Error $e = \frac{1}{2x}$
One	1—9	½ to 1/18, or 50% to 5.5%
Two	10—99	1/20 to 1/198, or 5% to 0.505%
Three	100—999	1/200 to 1/1998, or 0.5% to 0.050%

Thus, a measured result which is "correct to two significant figures" has a percentage error varying between 5% and 0.505%.

Standards of Accuracy. What shall constitute the standard of desirable accuracy in any given case? In some work a comparatively large error may safely be permitted, while in other work such an error would be fatal. The table on page 189 may be used to check the accuracy obtained in any measured result.[27]

[25] Tuttle and Satterly, *The Theory of Measurements*, p. 39.

[26] The writer is indebted to the text of Gale and Watkeys for this formula and the table based on it.

[27] See Tuttle and Satterly, *op. cit.*, p. 145.

Size of Errors [from Tuttle & Satterly]

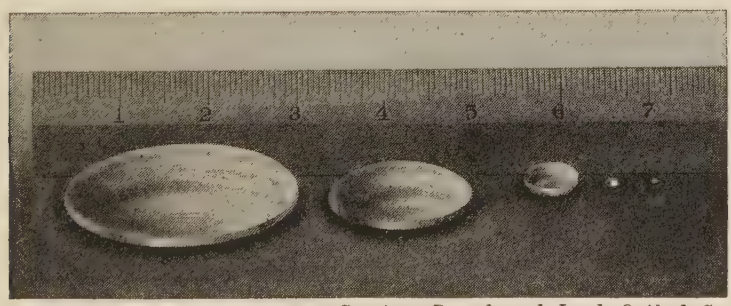
0.03 to 0.1 per cent	"very small"
0.1 to 0.3 per cent	"small"
0.3 to 1 per cent	"moderate"
1 to 3 per cent	"large"
3 to 10 per cent	"very large"

There is such a thing as superfluous accuracy. A knowledge of the degree of accuracy that is required in particular cases often makes it possible to save time and energy.

For the majority of engineering and commercial calculations three-figure accuracy is sufficient. Most of the calculations arising in physics and chemistry do not demand more than four significant figures. Some of the calculations necessary in surveying and in astronomy require six or even seven significant figures.

In carpentry, measurements are made to 1/16 inch, to 1/32 inch, and in special cases to 1/64 inch. Machinists' steel scales are graduated to 1/64 inch and 1/100 inch. Careful machinists can often measure to 1/1000 inch with such a scale, 5/1000 inch being readily obtained.

As a stimulus to accuracy, however, and for informational purposes, it is well to acquaint pupils with particular instances of precision work, preferably derived from local industries. Samples of high-grade machine work or pictures illustrating precision methods are particularly valuable. Thus, the accompanying photograph



Courtesy Bausch and Lomb Optical Co.

Fig. 8

shows some high-grade lenses. The small hemispherical lens at the right is but three-hundredths of an inch in diameter and is accurate in dimensions to ten-thousandths, while the errors in the surfaces are reduced to millionths of an inch. Tool microscopes measure with an accuracy of 0.0001 inch. In polishing the surfaces

of prisms used in range finders, the surfaces do not vary from a true plane by more than about a millionth of an inch.[27]

Similar remarks hold with reference to such standard gages as those manufactured by the Johansson Gage Company of Poughkeepsie and Detroit. These gages will measure to one-millionth of an inch.[28] The apparatus used by Professor Michelson, in determining the meter in terms of wave-lengths of light, represented the highest skill of the instrument maker. The mirrors and optical planes in some cases reached an accuracy as great as 1/40,000 of a millimeter, or the 1/20 part of the average wave-length of light. In repeating the Michelson-Morley ether-drift experiment of 1887, Professor Miller of Cleveland recently set up an interferometer apparatus near the Mt. Wilson Observatory in California, by which it was possible to detect a variation of one part in 100,000,000.

In surveying, too, a relatively high degree of accuracy is now possible. "If an accurate steel tape is used, which is stretched by a measured force when it is at a carefully determined temperature, it is possible in the course of a few weeks to measure a base line for surveying purposes with an error of about one unit out of 500,000." (Tuttle, p. 172.)

Schoolroom Limits of Accuracy. In comparison with the high standards mentioned above, the classroom must be satisfied with very rough approximations. For the purposes of everyday life, such approximations usually represent a sufficient degree of accuracy. An ordinary ruler, graduated to eighths or sixteenths of an inch, automatically imposes certain limits on the possible degree of accuracy. Thus, if in measuring a length of one foot, a pupil made an error of 1/8 inch, his relative error would be 1/96, or nearly 1%. This is "moderately" accurate work. If the width of a schoolroom is 20 feet, and is measured with an error of 1/4 inch, the percentage of error is readily seen to be about 0.1 of 1%, which is a "small" error.

Again, if a pupil measures a line 20 centimeters long and makes an error of 1 millimeter, his percentage of error is 1/2 of 1%, a "moderate" error.

[27] The photograph and the explanatory data are printed through the courtesy of the Bausch and Lomb Optical Company, Rochester, N. Y.

[28] Valuable articles on accurate measurement will be found in the technical journals, such as *Machinery* and *Mechanical Engineering*. See especially a discussion of optical methods of measuring machine parts, by Henry F. Kurtz of the Scientific Bureau of the Bausch and Lomb Optical Company, in the Mid-November number, 1925, of *Mechanical Engineering*.

These simple illustrations may serve to indicate the type of measurement work and the degree of accuracy which is possible or desirable in the average schoolroom.

Approximate Computations. Since every measurement involves an error, it is evident that computations based on these approximate measurements (for example, perimeters, areas, volumes) are also more or less in error, no matter how accurately the computations are made. Hence, if only rough approximations are obtained in measuring, it would be absurd to expect a higher degree of accuracy from the computations based on them. This fact has led to a technique of abridged multiplication and division of decimals, which is very valuable to those who have occasion to use it. It is not as difficult to understand as some teachers think.

It is doubtful, however, whether time can be found in the junior high school for the teaching of these methods. Survey tests have shown that many pupils in the junior high schools are still very deficient in their mastery of the fundamental arithmetical skills. Careful experiments seem to indicate that abbreviated multiplication and division of decimals can be taught effectively in the ninth grade, but that these processes are not retained very long unless a disproportionate amount of attention is given to them. Instead, we may have to be satisfied for the present with a more modest program. Pupils can be taught to "round off" results. They can also understand the idea that a chain is not stronger than its weakest link. Hence, in general, all results in a computation should be rounded off to correspond to the *least* reliable element. At the present time too little emphasis is given to such work.

Thus, if the diameter of a circle is measured to the nearest tenth of an inch, say 21.7 inches, the product of 21.7 by 3.14159 gives a trustworthy result *only to the first decimal place.* In other words, the circumference cannot be computed correctly to any further percentage of accuracy than that with which the diameter is measured.

The actual calculation showing how the area of a square or of a circle is affected by making a specified assumed error in the necessary measurements represents a type of work which could be undertaken with profit by pupils of average ability.

For a detailed presentation of the theory underlying approximate computations, the reader is referred to the treatises previously mentioned. It may be of interest, however, to state here the prin-

ciples by which one may determine the possible error in a computation involving approximate measurements.[29]

1. The possible error of the *sum* of two or more measurements is equal to the sum of their individual possible errors.

2. The possible error of the *difference* between two measurements is equal to the sum (not the difference) of their possible errors.

3. The possible percentage error of a *product* is equal to the sum of the possible percentage errors of its factors.

4. The possible percentage error of a *quotient* is equal to the sum of the possible percentage errors of the divisor and dividend.

Summary and Conclusion

Aspects of Linear Measurement. In the preceding pages the attempt was made to outline some of the cultural, pedagogical, and technical aspects of linear measurement.

1. It has been shown that measurement is indeed the master art and that it should receive far more attention in our schools than is ordinarily given to it. The brief sketch of the gradual evolution of our standard units has perhaps emphasized sufficiently a chapter in human history which should become known to every child.

2. A simple, tested technique of linear measurement, suitable for seventh grade pupils of ordinary ability, has been submitted. It is not perfect and can certainly be improved by any progressive teacher. Standards of accuracy were given. Above all, it was indicated what is the present classroom limit of possible or desirable accuracy.

If an apology be considered necessary for the extensive treatment of linear measurement presented in this report, it may be pointed out again that this phase of measurement is basic. It is not only of the highest importance from a cultural point of view, but it is also indispensable as a foundation for an intelligent understanding of the mensuration of plane and solid figures, for scale drawing, and for subsequent work in indirect measurement, including trigonometry. Nevertheless, it would be incorrect to infer that the total time to be allotted to measurement, as outlined above, is to be unduly liberal. On the contrary, a few minutes of carefully guided work each day during a total period of three or four weeks will be

[29] Tuttle, *op. cit.*, p. 176; Gale and Watkeys, *op. cit.*, p. 74; Schultze and Breckenridge, *op. cit.*, p. 330; Karpinski, Benedict and Calhoun, *op. cit.*, pp. 31-37.

a sufficient allowance during the seventh grade. If the doctrine is sound that computation and measurement are the two fundamental skills of elementary mathematics, they should be stressed continuously until a reasonable mastery has been achieved. This will not be the case if the pupil's training in linear measurement is limited to a few desultory paragraphs in the textbook or restricted to a single year. It is well to recall that the most progressive secondary schools of Europe are now emphasizing geometric drawing and measurement during the entire secondary period.

Nor must it be supposed that computation with prescribed data can ever be a substitute for actual measurement. To ask pupils in the seventh grade, in the very first lessons, to "round off" decimals before they have had a definite contact with approximate measurements, seems pedagogically useless and unsound. Accurate measurement in the classroom cannot be carried very much beyond the limits indicated above. It follows as a corollary that all computations based on measurement should be limited, for a considerable period of time, to the degree of accuracy which is actually attainable with the customary instruments.

3. Finally, even the technical phases of measurement were seen to have an important cultural message. Precision of measurement underlies our entire industrial era. Each of us is the beneficiary, in a thousand unsuspected ways, of the art of exact measurement.

More than that, this marvelous art is the one bond that enables man to reach out into the infinity of space and thus become oriented in his Larger Home. When it had become possible to define the meter in terms of wave-lengths of light, an epoch-making feat had been accomplished. For, "if man were transported to the uttermost confines of the universe, he would still find there the same little waves of light, and they would be just the same as here."

"If some day we are able to communicate with the dwellers upon some other planet, it will be a simple thing to communicate to them our standard of length and time and mass, and with the little waves of light which convey our message we may ultimately impart our exact knowledge to them, and receive theirs in return. The laws of light, of motion, of gravitation, of electricity, are undoubtedly identical for the whole universe, and given the first communication of another world we should be able to establish a truly universal system of units and standards. By this means inter-planetary communication would be placed upon a quantitative basis, and the

omnipresent, everlasting, but ultra-microscopic wave of light would be the universal unchanging standard."

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THE USE OF MEASURING INSTRUMENTS IN TEACHING MATHEMATICS

By C. N. SHUSTER
Orange High School, Orange, N. J.

Importance of Instruments. In the days when many teachers considered "factoring eighty per cent of algebra" and when the old formal propositions of geometry with few originals or practical applications held sway, a straightedge and a compass were sufficient for all our needs. Today, however, with our less formal junior high school mathematics, general mathematics, greatly increased number of practical applications, early introduction of numerical trigonometry, and realization of our obligations to the students who will not go to college, the need of the various measuring instruments and other tools of mathematics becomes so imperative that it is difficult to see how satisfactory work can be done without them.

The use of instruments greatly enhances interest in mathematics, and makes it seem more real and practical. They put the tang of sea and the spirit of scouting into our work. It is a wonderful change to get away from the text and out into the woods, where we can make a map of the camp, find the height of a tall tree with a crow's nest in it, and measure the length of a lake, or the width of a river. Even if we cannot leave the classroom, we can put an unbelievable amount of enthusiasm and interest into our work by the use of a sextant, an angle mirror, or some other such instrument. In these days of large high school attendance when we do not always secure the grade of intelligence that once obtained, we cannot afford to neglect any appeal to interest.

Many constructions and propositions that formerly seemed dry and useless take on new meaning as a result of this practical field work. A good example is the problem "Upon a given line as a chord construct a segment of a circle in which a given angle can be inscribed." Several interesting constructions can be given in solving this problem and its proof involves good geometry; but it

is on the point of being discarded. As a matter of fact, however, this problem is much used in navigation. If there is a dangerous reef, a submerged rock, or other obstruction near the shore, the danger angle can be given and the navigator can avoid the rocks by making use of his knowledge of mathematics.

Fig. 1 shows how the danger angle is used. In this case a double danger angle is given. A ship staying between these angles would pass between the obstructions. If you wish to enliven mathematics, have your pupils make a sextant and "navigate" a wagon between obstructions known to the judges but not to the navigator. Even

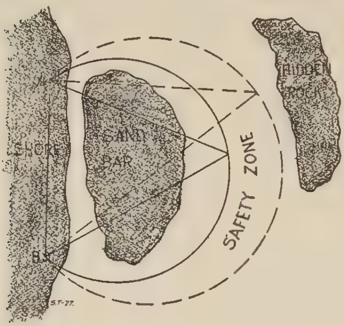


Fig. 1

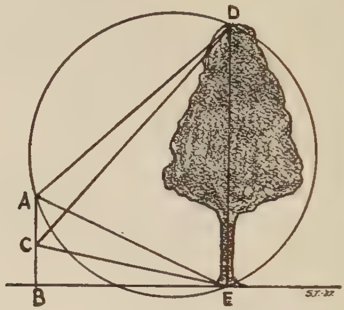


Fig. 2

graduate students in college would enjoy doing this. Once a very tall girl was working with a short girl as a partner. They were using a sextant to measure the angle subtended by a tree. They both stood on the same spot, but their results did not agree. The instructor put a drawing like Fig. 2 on the board to explain the difficulty. When the distance from the tree is such that the angle subtended by the tree is 90°, the sextant is on a circle having the tree for a diameter. If the distance to the tree and the distance of the sextant above the ground are measured, the diameter or height of the tree can be found by using the fact that the perpendicular let fall from any point on the circle to the diameter is the mean proportional between the segments of the diameter.

Again, let us suppose that we have partly mapped a certain tract with a plane table and wish to "set up" at a new point. This point we must locate on the map. We set up the plane table over the new point, place a blank sheet of paper on our board, and sight *A, B,* and *C* the known points. This gives us angles x and y. (Fig. 3.) Now, we take our map, and on AB and BC as chords

construct arcs of circles in which angles x and y may be inscribed. Point P will be at the place where these circles intersect. (Fig. 4.)

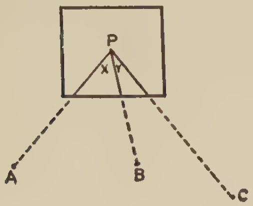


Fig. 3

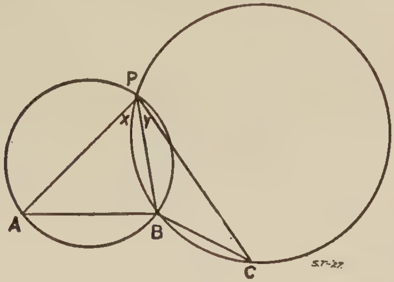


Fig. 4

This method is also used by navigators to locate a ship when near the coast. Finally, if we wish to construct a large circle in a park or other place where size or obstructions prevent us from using the radius, we may erect two poles and with the help of a sextant locate the circle as in Fig. 1. A picture of a sextant is given in Fig. 5. After studying these exercises almost any class would want to retain this problem.

Prevocational and Vocational Value of Instruments. There is considerable prevocational value in the use of measuring instruments. These are the working tools of the surveyor, the engineer, the builder, the forester, the map maker, the navigator, and a host of other outdoor men who make their living through the applications of mathematics. Many a student may be influenced, through these wholesome contacts, to maintain a permanent interest in mathematics.

In addition to the prevocational value derived from the use of instruments, there is vocational training of considerable value. I have known a number of students who were able to get Saturday or summer positions with surveyors or engineers because of a knowledge of the use and care of instruments. Such experience certainly will be of value to the boy who goes to an engineering school. Moreover, many of our pupils who never go to college will profit greatly through such a course. Thousands of levels and transits retailing at from twenty-five to forty dollars are being sold to farmers and builders, and almost any man at some time or other needs to do a piece of work in which a knowledge of the use of the instruments of mathematics would be of value. Only recently

I was visiting a friend, in whose cellar a considerable amount of water had accumulated because of heavy rains. He accordingly decided to dig a trench from the cellar to a creek about 350 ft. distant and put in a tile drain. This he estimated would cost between $125 and $150. I suggested that it might be a good plan to find the difference in elevation between the bottom of his cellar and the creek. We did this using an ordinary carpenter's level costing 50¢, two 10 ft. poles, and a 16 ft. scantling. One pole was placed in the creek and held plumb. A pin was placed in this pole 2 ft. from the top and one end of the 16 ft. scantling placed on it. The carpenter's level was then placed on the scantling and when it was level a pin was inserted in the other 10 ft. pole under the end of the scantling. The first pole was then taken to a point 16 ft. from the second pole, the scantling placed on the second pin, and the same process repeated until the house was reached. The horizontal line thus extended from 8 ft. above the creek level and intersected the last pole at a point 3.5 ft. above the ground. This point was therefore 4.5 ft. above the level of the creek. As the floor of the cellar was 6 ft. below the surface of the ground it was actually 1.5 ft. below the level of the creek! It seems to me that any training that would enable a boy to do a similar job intelligently would be of value.

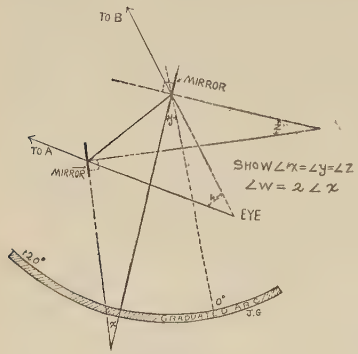


FIG. 5

Early Instruments. The roads, sewers, canals, aqueducts, and buildings of earlier civilizations arouse the admiration of our modern engineers. Yet all this work, together with the navigation, surveying, and astronomy of that time was accomplished with very simple instruments. Many of these are easy to construct, easy to use, involve valuable geometrical principles, and are interesting in that they are the ancestors of many of our modern measuring instruments.

Some of the most interesting early instruments are those used to measure or lay off right angles. They include the applying of the 6, 8, 10 relation by using a rope stretched taut. Fig. 6 shows how modern engineers still use this principle to lay off a right angle

with a steel tape. Figs. 7 and 8 show how an instrument similar to the modern carpenter's square was used. A right angle was laid off by sighting along AB and then along CD. (Figs. 9 and 10 show

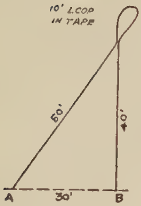


Fig. 6

Fig. 7

Fig. 8

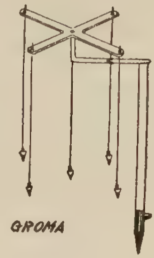


Fig. 9

the groma and surveyor's cross.) A modern form of these instruments is still used. Another interesting instrument was the cross-staff. Figs. 11 and 12 show its construction and use. The proof of the rule given involves some very fine geometry.

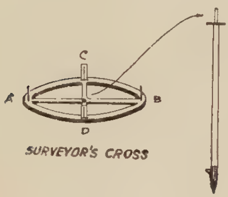


Fig. 10

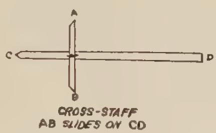


Fig. 11

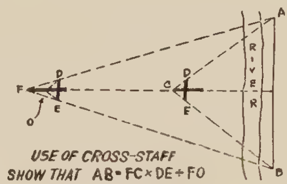
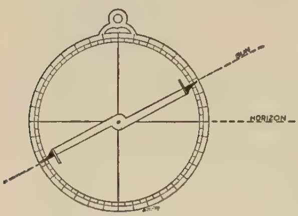


Fig. 12

The astrolabe (Fig. 13) was one of the most useful of all the early instruments. It was used to measure angles and thus did the work that we now do with a sextant or transit. In its simplest form it is a very easy instrument to construct. Take a large, circular bristol board protractor twelve to fourteen inches in diameter which is divided into quarter degrees. Tack it on a board and give it several coats of varnish and wax; then fit a ring in the top and a sighting arm or alidade as shown in this picture. If a staff is fitted to the back so it can be used in a horizontal position, it is possible to measure both vertical and horizontal angles, find the altitude of the sun, and do many other interesting things.

Another very useful instrument of the Middle Ages was the geometric square. (A quadrant was frequently attached also.)

This was quite similar to our modern hypsometers or clinometers so it is not necessary to describe it here. Davies' *Practical Mathematics* published as late as 1852 gives a full account of the geometric square as a practical instrument and a number of problems. The early level also was somewhat similar to a gravity clinometer. A fuller account of these early instruments will be found in Professor D. E. Smith's *History of Mathematics,* Vol. II, pp. 344-368.



THE ASTROLABE

FIG. 13

The question is often raised as to the accuracy of these early instruments or of the homemade models we now make. It is well to remember that these instruments have sights like those on a rifle. If a rifle is well made and the operator is skillful a high degree of accuracy is obtained. If, however, we desire any real precision we must use instruments equipped with a vernier and a telescope.

The Vernier. The vernier was invented by Pierre Vernier in 1631. It consists of a small sliding scale parallel to or concentric with a larger scale. With it we can measure fractional parts of one of the smallest divisions on the main scale. The best way to understand a vernier is to make and use one. Take a stick 1 in. by 1 in. and several feet long, and divide each foot into tenths. Then attach a piece of the same material so it will slide along the longer stick and divide a space on this equal to $\frac{9}{10}$ of a foot into 10 equal parts. Each of these parts will be 0.9 ft. ÷ 10 or 0.09 ft., or they will be 0.01 ft. shorter than the divisions on the main scale. Now slide the vernier along the fixed scale, starting, for simplicity sake, with an even foot. When the vernier is 0.01 past the starting point the first division of the vernier will be in line with the first division on the fixed scale, and so on. The vernier in Fig. 14 reads 7.05. A vernier can be made in the same way to read 0.01 in. or 0.001 ft.

Protractors may also be fitted with a vernier. This enables us to read angles to minutes or even to seconds. These verniers are found on vernier protractors, levels, transits, and sextants. A large vernier of this type should be made for class work. Take a large bristol board protractor at least 14 in. in diameter divided into degrees and half degrees, and place it on a larger piece of cardboard. Put a pin through the center so that the protractor or board may

be revolved. Then, on the cardboard, construct the vernier arc concentric with the protractor arc and in contact with it. To construct take 29 of the half degree spaces and subdivide this arc into 30 parts. Each of the vernier divisions will be equal to $^{29}/_{30}$ of 30 minutes or 29 minutes. The difference between one division on the circle and one division on the vernier will be equal to one minute and the vernier will therefore read to minutes. This is the common type of vernier found on transits. Since the transit reads in either direction the vernier on a transit is usually double. The vernier

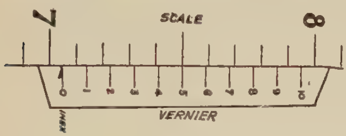


Fig. 14

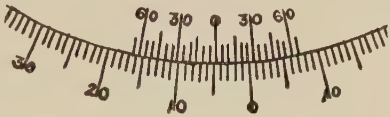


Fig. 15

may be placed either inside or outside the scale. The vernier in Fig. 15 reads 4 degrees and 45 minutes.

Pupils should be taught to read angles before using the instrument. A very good demonstration vernier can be made by taking a cut from an instrument catalogue and enlarging it with a pair of dividers. This is easier to make than the one described above. Demonstration verniers may also be purchased for a rather small price from the makers of instruments.

It is necessary to measure angles to the nearest minute only when our lines show four significant figures. Pupils should be impressed with the fact that all data which have been found by measuring are only approximate. All measurements on any piece of work should be made with the same degree of accuracy. Pupils are often inclined to carry computed measures to a refinement not warranted by the original measured data.

The number 4 shows one significant figure while the number 4.000 shows four, or it shows that the measure was taken to the nearest thousandth. The numbers 68.43 and 6843000 each have four significant figures. The number 6843000.0 has eight significant figures.

In "rounding off" numbers that have too many decimal places drop any digit less than five, and increase by 1 the digits preceding 6, 7, 8, and 9. The digit 5 is usually rounded off to an even digit thus: $6.843 = 6.84$, $6.846 = 6.85$, $6.845 = 6.84$, $6.835 = 6.84$. The rule for the number 5 will introduce compensating errors.

Methods of Locating a Point. One of the first steps in field work should be the location of a point with respect to at least two other points. As a matter of fact, this is one of the chief purposes of surveying. The several methods given in Fig. 16 should be learned. Discuss in class the geometric principle back of each method and the relative merits and uses of each, and have the pupils

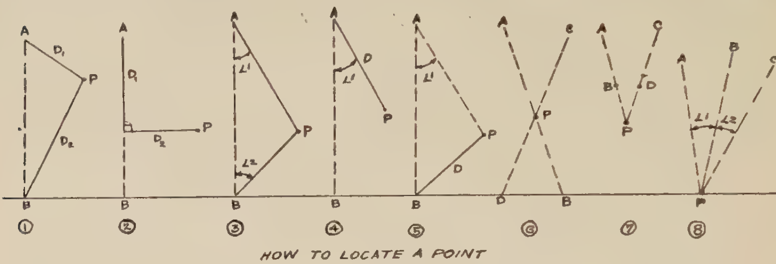


FIG. 16

locate points in the field by each method. With younger pupils interest can be aroused by hiding some object and having the pupils find it by the various methods given. Extra credit may be given to the groups that make the most finds. A regular treasure hunt is in order. In this connection the pupils will be interested in reading Poe's "Gold Bug."

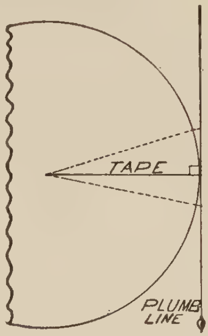
Linear Measurement. The most important and fundamental work in surveying or field mathematics is accurate linear measurement. It is also much more difficult than most persons would think. Ask eight or ten individuals to measure a distance of 1000 ft. over rolling land, and the truth of this statement will be shown.

Originally most distances were measured with a chain of 100 links. This chain was 4 rd. or 66 ft. long. The steel tape has now taken the place of the chain, but we still speak of chaining and call the men who measure with a tape chainmen. The steel tape is usually 100 ft. long. Other lengths are 25, 50, 150, and 200 ft. Tapes 15 and 30 meters long are coming into use. Our Government uses meters in place of feet for much of the work of the coast and geodetic survey. The 100 ft. tape may be marked in inches or in tenths of a foot.

The steel tape is easily broken by pulling on it when there is a kink or loop in it. It may also be damaged by walking over it, bending it around sharp corners, or dropping things on it. It also

rusts quickly and it is easy to break it out of the case or to break off the ends. It is very annoying to work with a broken or mended tape and as tapes are expensive they should always be handled carefully.

The two men measuring with a tape are called the head and rear chainmen. In chaining on a slope it is the horizontal distance that is desired, therefore the tape either must be held horizontally or the distance along the slope found and this corrected by calculation to get the true distance. In chaining down a slope when the tape is kept horizontal, a plumb bob is used to locate the point under the downhill end of the tape. If the tape is held in one hand and the plumb line in the other, it will be found that as the tape is moved up and down there is one position in which it is shortest. This is the horizontal position. The figure at the right shows why this is so.



In measuring a long distance it is easy to make an error of $\frac{1}{8}$ in. in marking tape lengths. Since $\frac{1}{8}$ in. equals about 0.01 ft. this would amount to 0.01 ft. for each 100 ft. measured or 0.04 ft. per 100 ft. if a 25 ft. tape were used. There is a change of about 0.01 ft. per 100 ft. of length in a steel tape for every 15° F. change in temperature. There is almost always a certain amount of sag in a tape, which produces a small error. To overcome this we try to put a certain standard pull on the tape. This, however, stretches the tape about 0.01 ft. for every 15 lb. pull and amateurs may "seesaw" and introduce a larger error than they seek to correct. It is hard to keep the tape in perfect alignment, and this introduces an error. It is hard to keep the tape level, or horizontal. There are also a number of other sources of error which we cannot enumerate here.

All the above sources of error will be quite discouraging to a person accustomed to using $\pi = 3.1416$, since most of them affect the results in the second decimal place. With a little care, however, satisfactory results can be obtained. Many of the errors cited are compensating. Experienced chainmen will often have an error of less than 0.01 ft. for 1000 ft. High school students may consider their work satisfactory even if they have an error as large as 0.1 ft. to 0.2 ft. in 1000 ft. or less than a foot in a mile. The fol-

lowing make interesting problems: (1) In measuring the distance between two points with a steel tape 100 ft. long A and C are on the true line but B is 1 ft. out of line. $AB = 100$ ft., $BC = 100$ ft., BD, the perpendicular from B to $AC = 1$ ft. Find the difference between AC the true distance and ABC the measured distance. (2) In chaining down hill the lower end of a 100 ft. tape is 1 ft. above or below the horizontal. What is the error in the horizontal distance?

A complete survey of a small tract may be made with a tape by establishing a network of triangles and measuring the three sides. The angles may then be computed, the tract mapped, and the area found.

A good exercise for using the tape is to stake out a rectangle for the foundation of a building. The method used is that given earlier in this article to get right angles. The fact that the diagonals of a rectangle are equal may be used as a check. The following problem taken from the Wentworth-Smith *Trigonometry* shows what can be done with only a tape: "To find the distance of an inaccessible point C from either of two points A and B, having no instruments to measure angles. Prolong CA to a, and CB to b, and draw AB, Ab, Ba. Measure AB, 500 ft.; aA, 100 ft.; aB, 560 ft.; bB, 100 ft.; and Ab, 550 ft. Compute the distances AC and BC." The answers are 536.3 ft. and 500.2 ft.

There are more than ten methods similar to the one used in the above problem that may be used in the same or similar situations. With the tape we can erect perpendiculars, draw parallel lines, construct an angle equal to given angles, lay off or measure angles by trigonometric functions, extend lines through obstacles, and perform a great number of other feats.

The Angle Mirror. The angle mirror or angle prism is an interesting little instrument that is inexpensive to buy or very easy to construct (one can be made at a cost of fifty cents). It is easy to learn to manipulate it and it has some interesting geometry as a foundation for its construction and use. This instrument, like many others, may be used by both the mathematics and the science departments. Fig. 17 shows the method of using the mirror. Fig. 18 shows how the mirror brings together objects that may be hundreds of feet apart and at an angle of 90° with the observer. This is one of the few remaining legal ways of "seeing double." The sextant works in the same way, so this sketch will do for the

sextant also. When the pole and the tree "line up," the angle formed by the tree, the angle mirror, and the pole is 90 degrees.

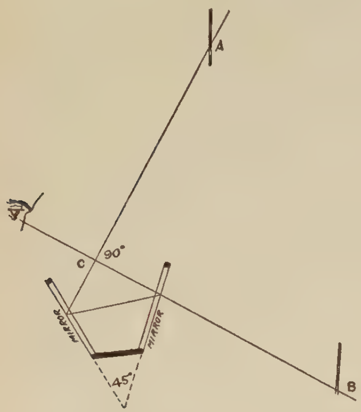


FIG. 17

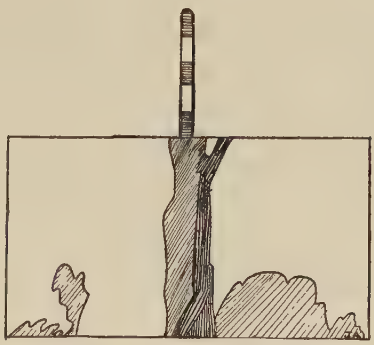


FIG. 18

Fig. 19 shows an exercise that can be performed with an angle mirror, a tape, and a pole. It is desired to sketch the bank of a creek or some other irregular line. Lay off a straight AB some distance from the creek and mark every 10 ft. (or some other interval). Then place a pole at B and have one student at A with an angle mirror while another student stands at the bank near C_1 with another pole. The student at A has the one at C_1 move until both poles line up. Angle C_1AB is then 90 degrees. The distance C_1A is measured and noted. Fig. 19 shows an actual exercise in which the student used a 50 ft. tape. Two students did the work. As soon as all the offsets were measured it was easy to plot the map. The creek was drawn in connecting C_1, C_2, and so on. The area between the line AB and the creek may be found by using the trapezoidal rule: *Multiply the distance between the offsets by half the sum of the first and last offsets plus the sum of the other offsets.* This assumes that the figures are approximately trapezoids. The students should prove the rule given above. Other projects that make use of the angle mirror are the laying out of a football field or a tennis court. A map may be made by taking a central baseline running through the territory to be mapped and locating all

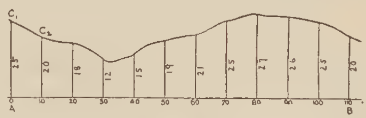


FIG. 19

the details by offsets from this line in much the same way as Fig. 19 was made. A number of the early instruments, such as the groma, may also be used for this kind of work.

The adjustable angle mirror is much more valuable for school work than the fixed angle mirror. The following statement quoted from the catalogue of the Keuffel and Esser Company shows the value of this instrument. "This Angle Mirror has the advantage that the angle of the mirrors is not fixed, but adjustable. It is determined by an arc graduated from zero to 100 degrees, figured in accordance with the angle of the sighted point, being consequently double the angle of the mirrors. With this instrument offsets may be laid down at any angle up to 100 degrees from a given base, and distances to inaccessible points may be determined by measuring base and angle, when distance = base × tangent of angle. This computation for distance can also be worked out in a very simple manner by means of the slide rule.

"This angle mirror will be found very useful, not only for the Surveyor and Civil Engineer, but also for the Military Officer, Traveler, etc."

This instrument, which is in many ways a little sextant, is also inexpensive and may be easily constructed by adding a protractor and hinge to a homemade fixed angle mirror.

Prism range finders used by military officers, surveyors, and explorers work on the same principle as that of the angle mirror and are interesting little instruments.

The Hypsometer. The hypsometer is the modern form of the geometric square. It is used to measure heights. It can easily be constructed by pasting a sheet of ordinary graph paper on a board and having a plumb bob and sights fitted. Varnish will help to protect the paper. Fig. 20 shows the construction, the geometric principles involved, and the method of using the hypsometer.

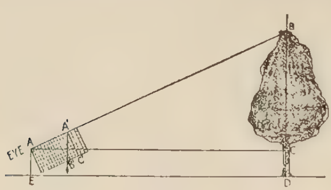


Fig. 20

Triangle $A'B'C'$ is similar to triangle ABC, so if we take distance on $A'C'$ (to scale) equal to AC and then go out one of the cross lines (corresponding to $B'C'$) to where the plumb bob cuts, that distance will be the height of tree. With a little practice a high degree of accuracy can be obtained. To aid in sighting a tube

with cross hairs may be fastened to the board or sights similar to those on a rifle attached. The advantages of the hypsometer are the speed with which observations may be made and the freedom from computing. Fig. 20 forms another good geometry original.

The Clinometer. The clinometer is an instrument closely related to the hypsometer and the early measuring instruments. If we take a half circle protractor, tack it upside down on a board 4 to 6 in. wide and 1 to 2 ft. long and attach a watch hand so it is free to swing by gravity, we have a first-class clinometer. When the instrument is level, the hand should stand at the center of the arc. If we get a protractor without numbers, this position should be marked zero and the protractor graduated up to 90° on each side. This may be used for leveling, for placing on any uniform slope to find the angle of slope, and for drawing lines making any angle with the horizontal on a blackboard, or it may be sighted like a hypsometer and angles of elevation or depression measured. It is a very handy instrument to have around a physics classroom. It may be used to measure heights by a method similar to that used with a hypsometer except that here we get an angle and a line and must compute the height.

Commercial clinometers are often fitted with a vernier. Some are also so constructed that the percentage of slope instead of an angle is indicated.

The Heliotrope. The heliotrope is another interesting instrument. Fig. 21 shows one that can be made in any shop. With it sunlight signals may be flashed for a distance of 20 mi. The hole nearer the mirror should be larger than the one at the end. This will be of value in scout work. Any code may be used. This instrument is frequently used by surveyors, foresters, and others who wish to communicate at a distance.

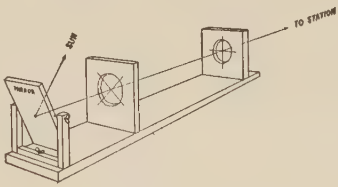


Fig. 21

The Transit. The transit, the most important of all measuring instruments, is simply a field protractor for measuring horizontal and, usually, vertical angles. The higher priced instruments are so accurate that it is possible to measure angles to ten seconds or less. Transits costing from $150 to $200 usually measure to one minute of arc. These are accurate enough for any school purpose. A

transit that measures to degrees and costs from $20 to $40 has recently been placed on the market. This is a very valuable instrument for schools that cannot afford a better instrument and is a splendid supplement to a higher priced transit. Students may learn to use first the cheaper instrument and finally the higher priced transit. It may also be used in junior high schools and elsewhere where a high degree of accuracy is not essential. A level that sells at from $15 to $25 is also available. A picture of a surveyor's transit will be found on page 221.

Since the transit will do all the work that can be done with a level, it is not necessary to have both instruments. The transit will do the work of most of the other instruments described in this chapter, but greater interest is aroused by a variety of instruments.

The transit may be used for a great variety of exercises that involve indirect measurements. Hundreds of these exercises can be found in our modern high school texts. The following are good types of these exercises:

1. Wishing to determine the distance between a church *A* and a tower *B*, on the opposite side of a river, a man measured a line *CD* along the river (*C* being nearly opposite *A*), and observed that the angles were respectively *ACB*, 58° 20′; *ACD*, 95° 20′; *ADB*, 53° 30′; *BDC*, 98° 45′. *CD* is 600 ft. What is the distance required? The answer is 757.5 ft.

2. To compute the horizontal distance between two inaccessible points *A* and *B* when no point can be found whence both can be seen. Take two points *C* and *D*, distance 200 yd., so that *A* can be seen from *C*, and *B* from *D*. From *C* measure *CF*, 200 yd. to *F*, whence *A* can be seen; and from *D* measure *DE*, 200 yd. to *E*, whence *B* can be seen. Measure *AFC*, 83°; *ACD*, 53° 30′; *ACF*, 54° 31′; *BDE*, 54° 30′; *BDC*, 156° 25′; *DEB*, 88° 30′. Compute the distance *AB*. The answer is 345.5 yd.

3. Wishing to find the height of summit *A*, a man measured a horizontal base line *CD* 440 yd. At *C* the angle of elevation of *A* is 37° 18′, and the horizontal angle between *D* and the summit of the mountain is 76° 18′; at *D* the horizontal angle between *C* and the summit is 67° 14′. Find the height. The answer is 520 yd.

4. A balloon is observed from two stations 3000 ft. apart. At the first station the horizontal angle of the balloon and the other station is 75° 25′, and the angle of elevation of the balloon is 18°. The horizontal angle of the first station and the balloon, measured at the second station, is 64° 30′. Find the height of the balloon. The answer is 1366 ft.

To understand these problems thoroughly and to secure interest in them all students should do a few of these problems in the field. Have students find the height of some local church spire or tall tree, the distance across a river, small lake, or swamp, the distance

between two points on the opposite sides of a hill, or continue a line where an obstacle prevents direct work, and so on. Field work will be of special value to students in trigonometry who have not

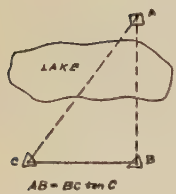


Fig. 22

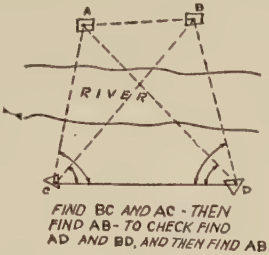


Fig. 23

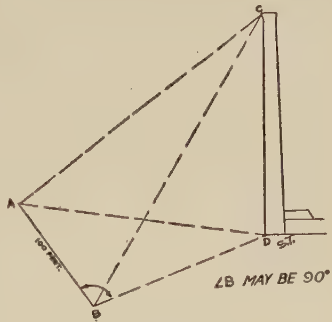


Fig. 24

studied solid geometry. Figs. 22, 23, and 24 show the use of the transit in a few of these exercises.

To Measure a Horizontal Angle. To measure a horizontal angle as XYZ with a transit:

1. Set the transit up with the plumb bob over Y.

2. Loosen both plates and revolve the plates until the zero of the vernier is opposite the zero on the limb. Then set the upper clamp and make the zeros exactly coincide by using the tangent screw.

3. With the plates clamped together sight on X. Set the lower clamp and with the lower tangent screw bring the line of sight exactly on X.

4. With the lower plate clamped, loosen the upper plate and with the upper plate turning sight Z. Clamp and get exact sight with the upper tangent screw.

5. Read the angle.

6. If it is desired to double or repeat the angle, leave the plates clamped, loosen the lower plate and sight X as before in (3). Then sight Z as in (4). The angle should now be double what it was in (5).

By repeating the angle several times we get a check and also a greater degree of accuracy. If an angle is repeated three times, divide by three. At first all angles should be repeated in order to give sufficient practice.

To measure vertical angles, or angles of depression or elevation, start with the telescope level and the vertical arc at zero. Sight the

object, set the clamp, and with the tangent screw sight the object exactly. Read the angle.

To measure the vertical angle between two objects sight one, read the angle, sight the other, and add or subtract as the case requires to get the angle.

It is very important to remember that transits, levels, sextants, and other precision instruments are very delicate, and must be handled with great care. They are made of soft metal, and get out of adjustment very easily. Never use force with any of these instruments. Carry a transit like a six-weeks-old infant and loosen clamps so the plates and telescope will turn if hit. (But don't let them be hit.)

Definitions. The following are definitions of terms used in the following pages:

Elevation (El.). The distance of a point above or below a level surface called the datum. This is usually mean sea-level. However, a datum is sometimes assumed.

Bench Mark (B.M.). A fixed point of known elevation that is used to find the elevation of other points.

Station. A point whose elevation is sought.

Plane of Sight. The horizontal plane determined by two sights through a properly adjusted level.

Backsight (B.S.). The reading of a rod on a point of known elevation. Its purpose is to determine the height of the instrument.

Height of Instrument. The elevation of the plane of sight.

Foresight (F.S.). The reading of the rod on a station whose elevation is sought.

Plane of Sight or H.I. The plane of sight or height of instrument should always exceed the elevation of the B.M. or F.S. for any high school work.

The following formulas are used:

$$\text{(El. of B.M.)} + \text{B.S.} = \text{H.I.} \qquad \text{H.I.} - \text{F.S.} = \text{El. (of S.)}$$

Backsights are sometimes called plus sights, and foresights, minus sights.

Profile Leveling. One of the best exercises with a level or transit is to make a profile level for an imaginary drain, and then make a map showing the traverse line (line surveyed) and some details. A profile of the traverse line or drain should be drawn as in Fig. 27.

As a first exercise select a piece of land with an interesting profile

and a gentle down grade. Let the drain be two lines at an angle of less than 120° with each other and each 100 or more feet long. Measure the lines, putting stakes every 25 ft. (or at any other regu-

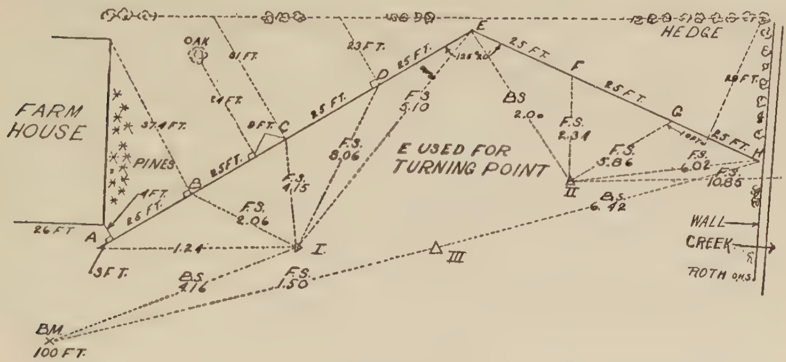


Fig. 25

lar interval). In doing this a student can line the chainmen (pupils with tape) by sighting along the line. This might be done with a transit for practice, but the former method will be found sufficiently accurate for most purposes. A sketch of such a line is shown in Fig. 25. Set the transit up over the point E and measure the angle E. The compass bearings of the lines AE and EH may be taken as a check or to give direction to the map. Fig. 26 shows how bearings and azimuths are read.

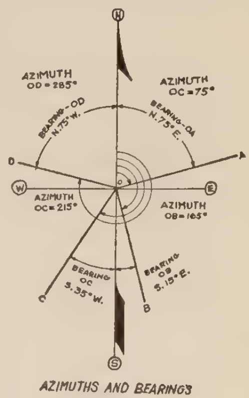


Fig. 26

Select some point near the proposed drain line for a bench mark and assume an elevation for it. (This need not be the true elevation since we desire only relative elevations.) Then set up the transit (or level) in some position from which the B.M. and a number of the stations can easily be seen. Take a backsight on the B.M. and foresights on the stations. The sketch shows a handy way to record the B.S.'s and the F.S.'s. It would, perhaps, be possible to take F.S.'s on all the stations from one set-up, but for practice require the pupils to make a second set-up. At each set-up the H.I. must be redetermined. To do this take a B.S. on any point whose elevation has previously been found. Such a point is

called a turning point (T.P.), which is really a temporary B.M. A point on the line may be used as a T.P. If an error is made at B.M. or T.P. it will affect all the elevations found with that H.I., so care should be taken. Take a B.S. on the T.P. and F.S.'s on all the other stations. Then, as a check, set up a third time and take a B.S. on the last station and a F.S. on the B.M. Thus the elevation of the B.M. is found. If an earthquake has not occurred since the B.S. was taken on the B.M., the assumed (or true) elevation and the one just found should agree. Any difference will be due to errors. These should never exceed $0.02\sqrt{\text{no. of set-ups}}$, and on a

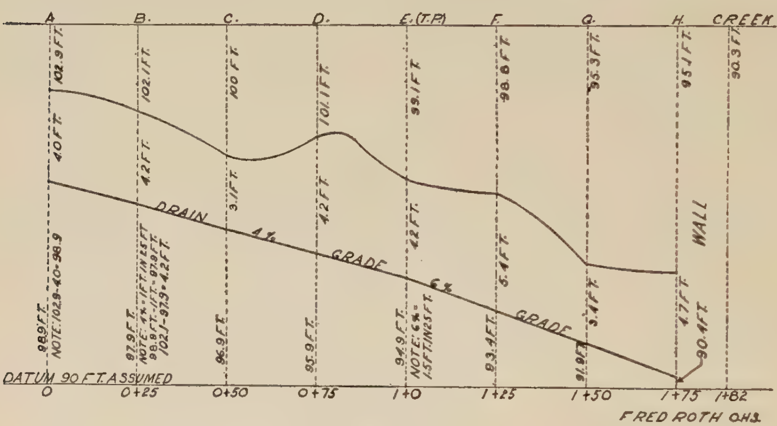


FIG. 27

small piece of work like this the error should not exceed 0.01 ft. On long sights curvature of the earth increases rod readings by an amount equal to 0.667 (distance in miles)^2 ft., while refraction tends to decrease readings by about $\frac{1}{7}$ of this; so if we correct for both by deducting 0.57 (distance in miles)^2 ft. where the instrument is at an equal distance from the points on which the B.S. and F.S. were taken, there will be no error. Why is this true? For short sights no attention should be paid to curvature and refraction.

The table (Fig. 28) shows how level notes may be recorded for permanent record. Use the formulas given to find the H.I. and the El.

To obtain detail for the map one group may use a plane table with the drain as a base line, or while the transit is set up over *E* angles may be measured, or any of the methods of locating a point shown in Fig. 16 may be used.

It is a good plan to give the pupils plenty of practice in reading the rod to three decimal places, but for ordinary practical work it is sufficient to read it to one or two decimal places. If the rod is resting on the ground, the slightest packing will change the elevation many thousands of a foot. The rod should always be held plumb. If, after a sight has been taken and the target fixed, the rod is waved to and from the transit, the target will sink. At its highest point the rod is vertical. Have pupils show why this is so; also have pupils read the rod when not plumb and again when plumb to show the error thus introduced.

STA.	B.S.(+)	H.I.	F.S.(−)	El	REMARKS
B.M.	4.16	104.16		100.00	El. ASSUMED
A.		104.16	1.24	102.92	IRON PIPE ABOUT 5 FT SOUTH OF HOUSE
B		104.16	2.06	102.10	
C		104.16	4.15	100.01	
D.		104.16	3.06	101.10	
E	2.06	101.12	5.10	99.06	IRON PIPE IN GROUND E USED FOR TURNING POINT
F.		101.12	2.34	98.78	
G.		101.12	5.86	95.26	
H.	6.42	101.52	6.02	95.10	NOTCH IN WALL CREEK F.S. 10.05
B.M.		101.52	1.50	100.02	CHECK ERROR .02
	FRED	ROTH			.02 √5 = .035

Fig. 28

Fig. 27 shows the profile of the drain. It is desirable to have the drain not less than 3 ft. below the surface to prevent freezing. If the drain is allowed to run much deeper than this, the amount of excavating is excessive. To prevent this it is sometimes necessary to change the gradient. The notes on Fig. 27 show how the various figures were obtained.

Triangulation. When it is desired to make an accurate and rather extended survey, an open level place is selected and a base line very accurately measured. From the ends of this line a point is sighted and the angles measured. The two sides of the triangle thus determined can be computed by trigonometry. These sides may in turn be used as base lines, and a network of triangles extended to cover the whole territory. This method of surveying is called triangulation. The best results will be obtained when the triangles are kept as nearly equilateral as possible. With students it is a good plan to measure a side now and then, and the last side should always be measured. The difference between the measured length of this side and its computed length will show the error in the survey. It will be found easier to measure angles accurately than to get a corresponding degree of accuracy with lines. With high school pupils it is not a good plan to try an extended survey by triangulation. A base line 100-300 ft. long with three or four triangles will give all the practice needed. Triangulation is the basis of geodetic surveying. Geodetic surveying differs from plane surveying in that the curvature of the earth's surface is

considered in the latter. The difference in length between a straight line and an arc on the surface of the earth is only about ½ in. in 10 miles.

Closed Traverse. When a farm or city lot is surveyed, the transit lines form a polygon. This is called a closed traverse. All the sides and angles are measured, and from these data we may plot the tract and find this area. Since the interior angles of a polygon are equal to $(n - 2)$ straight angles, we may use this fact as a check on the accuracy of our angles. Since there are always small errors in both the lines and the angles, there is always an error of closure. If, for high school work, we select a tract with from four to six sides, with the sides not more than a few hundred feet long, and if the work is rather carefully done, the error of closure will be negligible.

The Compass. The compass is one of the important instruments of navigation and is also extensively used by the forester, geologist, miner, explorer, hunter, guide, aviator, and military officer.

At one time, the surveyor's compass was the chief instrument used for surveying in country districts, and it is still used where speed rather than a high degree of accuracy is desired. The space devoted to various types of compasses in any large instrument catalogue eloquently attests the popularity of the compass. It has the following good points: (1) It is inexpensive. A good compass costs from $25 to $50. (2) It is lighter and much easier to carry than the transit. (3) It requires much less time to do a given job with a compass than with a transit. (4) As the direction of each line is determined by its magnetic bearings, a mistake in the direction of any one line will not affect every line, as is the case with the transit. (5) If, in running a line, we hit a house or some other obstacle, we may go on the other side and set up so that the backsight will have the same bearing as our previous foresight. This will be an extension of our former line, since only one line can have that bearing. This is simpler than the method of procedure with a transit.

Many compasses are equipped with a clinometer, which adds to their usefulness by making it possible to measure slope or read vertical angles.

A compass is mounted on a light tripod or Jacob staff. A compass may be used to advantage with a plane table. The vernier on a good compass usually reads to 5 minutes.

Magnetic Declination. The compass needle does not point to the true north pole but to the magnetic pole. The angle between true north-south meridian and the magnetic meridian is called the magnetic declination. Imaginary lines connecting places having the same declination are called isogonic lines. These lines are very irregular. The declination in the United States ranges from 22° west to 24° east. This declination unfortunately does not stay put over certain belts as does our time, but changes with each location and, worse yet, the declination of any one spot is constantly changing from hour to hour as well as from month to month, year to year, and age to age. Tables are given in most works on surveying that enable us to make the proper corrections for declination.

The trouble caused by magnetic declination, and the fact that bearings cannot be read closer than five minutes and that local attraction may be so great as to make the compass readings worthless are the chief disadvantages of the compass.

The Plane Table. The plane table is one of the most important instruments for general mapping. It is so simple that it may be

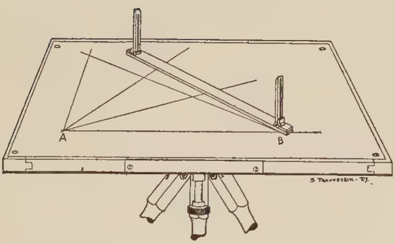


Fig. 29

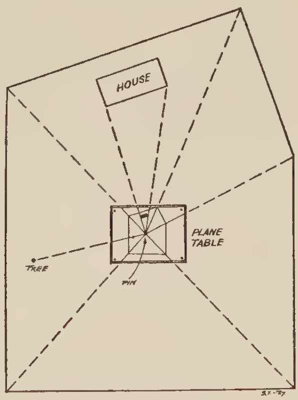


Fig. 30

used by a seventh grade class and so accurate that many of the best engineers prefer it to any other instrument for mapping.

The plane table is of value in high school work and for grade work in geography to show how a map is actually made, and it is by far the best instrument for scout work in mapping.

A simple plane table or traverse table outfit may be obtained

for about $25. Satisfactory work on a small scale can be done with a very crude homemade outfit. One pupil, using a stool, a drawing board, and a chalk box for an alidade, staged a contest with a civil engineer using a transit, and the resulting maps were so alike that it was hard to tell which were made with the expensive outfit. Of course only a small area was mapped in this contest.

Fig. 29 shows that the plane table consists of a tripod, drawing board, and a sighting instrument called an alidade.

With the plane table, all the important work on the map is done in the field. Angles are drawn directly on the map and not measured and then plotted. It is useless to measure an angle to 10 seconds if an error of 10 minutes is made in plotting the map.

On high-priced alidades, there is a telescope with stadia hairs. This aids in measuring distances with a fair degree of accuracy without using a tape.

To make a map of a small area, set up the plane table in the center of the territory to be mapped and clamp the board. Place a pin in the center of the table and with edge of the alidade against this pin sight all the points to be located and draw lines to them. Measure these distances with a tape or rod and lay them off on the map to scale. This will give an accurate location of these points on the map. This method is called radiation. (Fig. 30.)

For a larger area, measure off on the ground a straight base line *AB*, 100, 500, or 1000 ft. long, as desired, and draw to scale on the plane table a line *ab* to correspond to the line *AB*. Set up the plane table at *A* with point *a* approximately over *A*. Place the edge of the alidade along the line *ab* and sight *B*. Clamp the table in this position. Then, with the edge of the alidade at *a* sight all the points to be located and draw lines on the paper toward them. Then take the table to *B* and sight *A* similarly, so the lines *AB* and *ab* will have the same direction and *b* will be over *B*; clamp the table. Again sight and draw lines to the objects to be located as at *A*. Each object located on the map will be determined by the intersection of the lines from *a* and *b*. Some system of numbering the lines must be used to pre-

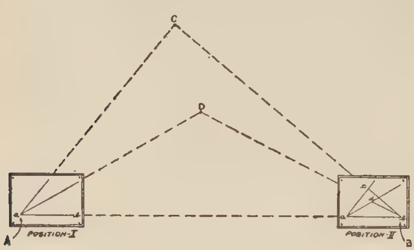


Fig. 31

vent confusion. The line AB may be extended and the work continued as before or some other line may be used. Radiation may also be used in connection with this method. This last method is called intersection and corresponds to triangulation. (Fig. 31.)

A solution of the three-point problem is given under danger angle. See Figs. 1 and 2.

A survey of a map may be made with a protractor for a transit and a scale for a tape and many interesting problems worked out or areas computed.

The plane table methods described above may also be used for making a small map from a larger map by using a sheet of paper for a plane table and a ruler for an alidade. In fact, to be sure that the methods to be used are understood, it would be a good plan to have the pupils do this before going out into the field. Much valuable time is wasted in field work because each member of the party does not know exactly what he is supposed to do, and the best and quickest method of doing it. Large geodetic survey maps are good for this purpose. Also, a good reduced copy of drawing or picture may be made in this way. This is a splendid exercise for a geography class. It should be noted that many of the other methods of mapping, as by off-sets, may be used for indoor class work of this type. Pupils should be encouraged to do some of this field work after school, on Saturdays, or during vacations. It is an interesting fact that just as girls are often as much interested in scouting as boys, so this work may also be of considerable interest to girls.

If the pupils in a school are divided into A, B, and C sections according to ability, much better field work can be done.

Stadia Surveying. This type of work is an interesting and practical application of similar triangles. With a fine hacksaw cut two notches $\frac{1}{8}$ in. apart through the ends of a brass tube $12\frac{1}{2}$ in.

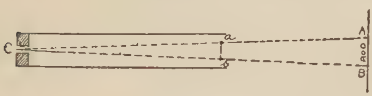


Fig. 32

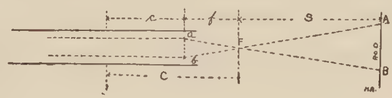


Fig. 33

long and $\frac{1}{2}$ in. in diameter. Put two fine black threads across the ends and fasten with a rubber band. Place a cork with a small hole in it in the other end of the tube.

Fig. 32 shows that triangles Cab and CAB are similar. Since the

perpendicular distance from C to ab is 100 times ab, the perpendicular distance from C to AB will be 100 times AB if AB is 1½ ft. The distance from C to the rod will be 150 ft. A stadia may be used with a hypsometer, a clinometer, or an alidade for approximate measuring. Transits and alidades with telescopes are usually equipped with stadia-hairs. With these the arrangement is like that in Fig. 33.

In most of these instruments $f \div ab = 100$ (Fig. 33), that is, when $AB = 1$ ft., the distance from the center of the instrument is $100 + C$ ft. C is usually about 1 ft. It may be found by setting up the transit on a level piece of ground and holding the rod at such a distance that the stadia-hairs intercept an interval of 1 ft. on the rod. If the distance is found to be 101.4 ft., the constant $C = 1.4$ ft. This must be added to all readings, as if $AB = 3$ ft., the distance will be $300 + 1.4$ ft. or 301.4 ft. With good instruments and careful work it is possible to have an error of less than 1 ft. in 500 ft.

If the country is hilly and the line of sight must be inclined, measurement by stadia will not be correct. Corrections may be worked out by trigonometry, or a slide rule, and all books on surveying have stadia reduction tables in which both the correct distance and the elevation may be found. If the stadia reading is 100 and the angle 10°, the horizontal distance will be 96.98 ft. and the difference in elevation 17.10 ft. Where the inclination is 5° or less, there is a horizontal difference of less than 1 ft. in 100 ft. For rough work we need not bother with angles less than 3°.

To find the horizontal distance and the vertical height from inclined stadia readings the following formulas are used:

Let

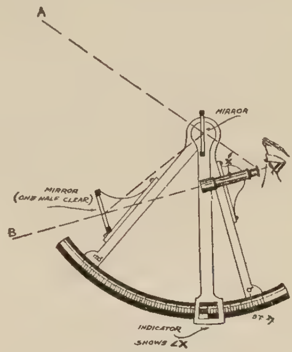
$$\text{R} = \text{actual reading}$$
$$\text{H.D.} = \text{true horizontal distance}$$
$$\text{V.H.} = \text{true vertical height}$$
$$x = \text{angle of inclination}$$

Then

$$\text{H.D.} = \text{R} \times \cos^2 x$$
$$\text{V.H.} = (\text{R} \times \sin 2x) \div 2$$

Stadia slide rules will solve these equations with one setting. As the stadia readings, the above formulas, and all functions are approximate, the slide rule will be found sufficiently accurate for this purpose.

The Sextant. The sextant (Fig. 34) is used to measure horizontal, vertical, or inclined angles with a high degree of accuracy. Moreover, this accuracy is not affected even though the instrument is in motion. Because of this, the sextant may be used on a ship or an aeroplane. It is also used by surveyors for work on land, and along shores, and for observing the sun or a star. The sextant has an appeal to students that makes it a very valuable instrument for school work. It can be used in dozens of interesting situations. One of its chief uses is to determine latitude. To do this with accuracy, corrections must be made for declination and also for refraction, parallax and semi-diameter of the sun. Corrections for the last three amount to about 15 min. in the United States and Canada. Corrections for declination are given in the *Nautical Almanac*. On March 21 and Sept. 23 there is no declination. Pupils may be asked to draw a sketch similar to Fig. 35 showing this fact. On these dates the pupils in any school may easily determine latitude by using a sextant, a transit, or an astrolabe (or measuring quadrant).



-THE SEXTANT-

FIG. 34

To Find the Altitude of the Sun. To find the altitude of the sun with a transit on these dates, set up the transit a few minutes

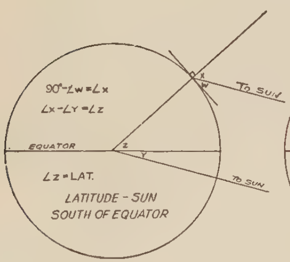


FIG. 35

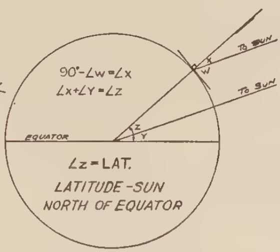


FIG. 36

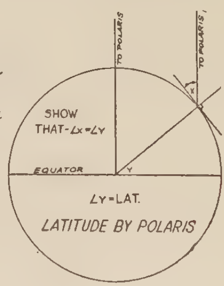


FIG. 37

before noon, with a dark glass over the object glass. Get the horizontal crosshair tangent to the lower edge of the sun and follow it up as long as it continues to rise. Then read the angle of elevation.

Add 15 min. to this angle and subtract the result from 90° to get the latitude. Check by getting the latitude from a map.

If an observation is taken from an elevated portion of a ship it is necessary to correct for the dip of the sea horizon, which amounts to 5 min. for an elevation of 26 ft. and 9 min. 48 sec. for 100 ft. This is subtracted from the observed altitude of the sun. The teacher will find it worthwhile to make a sketch and have advanced classes figure out the dip. It makes a good problem if a seven-place log table is available. Approximately, the dip in minutes is equal to the square root of the height in feet. On land the arti-

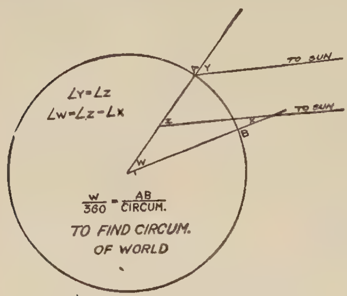


FIG. 38

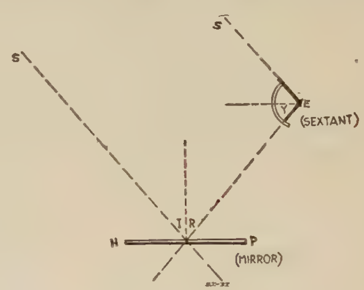
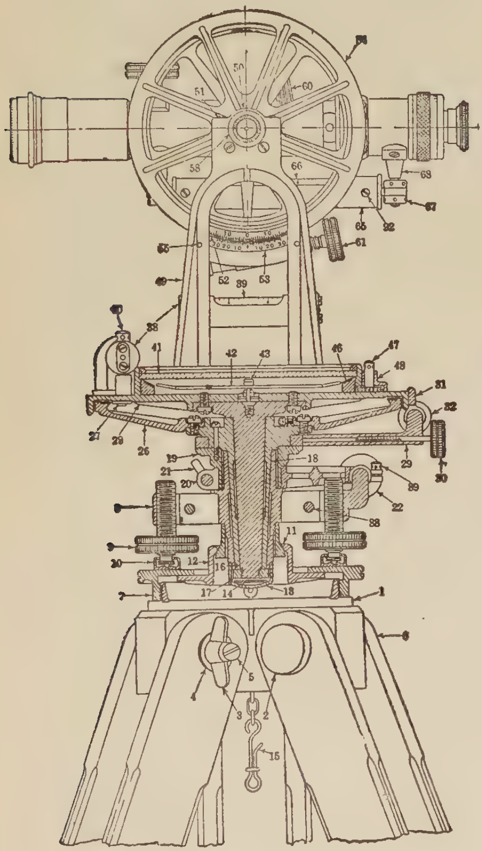


FIG. 39

ficial horizon is used with the sextant to find the altitude of the sun. Fig. 39 shows that the artificial horizon gives us another geometry original. An ordinary mirror resting on a level surface will serve for an artificial horizon. In this connection it may be noted that many early books on practical mathematics gave a method of finding the height of a house, a tree, or other object by means of a mirror. A mirror was placed at some convenient distance from the building or object. When the observer could see the top of the building in the mirror, he measured his distance from the mirror, the distance from his eyes to the ground, and the distance to the base of the building. As the angles at the mirror are equal, the triangles are similar; thus the height of the building can be computed. Figs. 35 to 38 show interesting exercises.

Determination of a Meridian. An exercise in field or laboratory mathematics that requires practically no apparatus is the determination of a meridian. Suspend a plumb bob over a flat level surface (cement area is best) so that the point of the bob is off the ground. Place a bead on the line some distance above the plumb bob. Some time in the morning mark the position of the bead's

shadow (S). Then from the point under the bob (P), and with a radius equal to PS draw a circle. At some time in the afternoon the shadow of the bead will again touch the circle. Mark this position D. Then bisect the angle DPS and the bisector will be the meridian. This may be determined with accuracy and permanently marked. More exact methods of determining the true meridian depend upon observations on Polaris. If the meridian is known, it is easy to determine the magnetic declination by using a compass.



Courtesy of Keuffel and Esser Co.

FIG. 40

1 Tripod head
2 " bolt
3 " " nut
4 " " washer
5 " " lock screw
6 " leg
7 " plate
8 Leveling head
9 " screw
10 " " shoe
11 Half ball
12 Shifting plate
13 Center cap
14 " spring
15 Plumb bob chain & hook
16 Inner center
17 Center nut
18 Outer center
19 Clamp collar
20 Lower clamp
21 " " screw
22 " " tangent screw
23 " " spring
24 " " plunger
25 " " cap
26 Horizontal limb
27 Vernier
28 " plate
29 " " clamp
30 " " " screw
31 " " " spring box
32 " " " tangent screw
38 " plate level
39 " " " vial
40 " " " adjusting screw
41 Compass cover glass
42 " needle
43 " " cap
46 " ring
47 " variation pinion
48 " " " housing
49 A-standard
50 " cap
51 " " screw
52 Vertical circle
53 " " vernier
54 " " guard
55 Vernier adjusting screw
58 Telescope axis end cap
60 " clamp screw
61 " " tangent screw
65 " level
66 " " vial
67 " " adjusting nut
68 " " support

The Slide Rule. This instrument is indispensable for performing the various calculations needed in connection with field work and for checking more accurate calculations made on paper. The slide rule will be found sufficiently accurate for work involving angles measured to 5 or 10 minutes and lines measured to three significant figures. In fact, the slide rule should be used in every trigonometry and physics class if not in every class in mathematics and science. Very satisfactory students' rules can be purchased for one dollar or less, so the element of cost is negligible.

The time that a student takes to master the slide rule is insignificant in comparison with the saving of time in later work.

Conclusion. It would be possible to teach physics, chemistry, or biology without apparatus and still do fairly good work. We can teach mathematics without instruments, but we can do much better with them. A good transit, similar to the one shown in Fig. 40, plane table outfit, sextant and angle mirror, together with the rods, tapes, and the like, which we need can be purchased for about $350. A cheaper set of instruments may be obtained for $150. Some of these instruments may be used in other departments, and with care will last from fifteen to twenty years or longer, so that the yearly cost is very small. If, for example, the use of instruments increased the size of a trigonometry class from 6 to 16 pupils, the saving in teacher cost would more than pay the cost of the instruments.

We must convince our superintendents and boards of education of the necessity of some instruments in teaching mathematics. We all know of schools with thousands of dollars worth of science apparatus and a mathematics equipment consisting of several old yardsticks and a couple of partly broken blackboard compasses.

As a last resort, if new instruments cannot be obtained, it may be possible to buy second-hand instruments no longer sufficiently accurate for engineering purposes, but still of value for school use. Homemade instruments may be used to supplement other instruments, to give variety, and to illustrate the interesting principles involved, but they cannot take the place of commercial instruments.

PROBLEM-SOLVING IN ARITHMETIC

By LUCIE L. DOWER
City Normal School, Rochester, N. Y.

Introduction. Mastery of the fundamental facts and processes is not the ultimate end of arithmetical instruction. Life demands that boys and girls have not only a perfect mastery of the fundamentals of arithmetic but the ability to interpret, comprehend, and solve the quantitative situations that arise in everyday activities. This preparation may be brought about in two ways:

1. By giving specific preparation for the kinds of problems the pupils will meet in life.
2. By giving general preparation for all kinds of problems.

If *specific* preparation for the kinds of problems that the pupils will meet in life were desired, it would be necessary to make a collection of all the types of problems available. The children could then be given a rule or a type solution for each of these types and drilled in its use. This method is out of the question, however, because conditions are constantly changing. It is not always possible to predict the kinds of problems which the pupils will have to solve. Furthermore, they would not be able to tell which of the many rules or type solutions to apply to a particular problem.

If, on the other hand, *general* preparation for all kinds of problems be given, the children will acquire the ability to judge a given problem on its own merits, to study the relationships that exist between the various quantities involved, and to think out the solution. This approach aims to develop skill in planning the solution as well as ability to execute the plan. This mastery of problem-solving comes only from meeting many different kinds of problems, from seeing many relations, and from reasoning out each problem in terms of the relationships that are involved. By this method, pupils are more likely to recognize similar problems in any new situation and to apply the correct solution.

The writer has prepared this chapter because general preparation

in problem-solving is so essential, and because the literature on problem-solving is not readily accessible. This presentation grew out of the studies made by the Arithmetic Research Committee in connection with the Coöperative School Survey which has been conducted in Rochester during the past two years. The report which follows is the writer's contribution to that Survey.*

Part One

Purpose. It is the purpose of the first part of this report to consider the major elements, facts, and principles essential in the formulation and the solution of desirable problems.

I. Sources of Problems

The Textbook as Source. Textbooks are often at fault in their large number of problems that involve adult activities and in their relatively small number of problems that concern the needs, activities, and interests of children. It is hopeless to expect children to be interested in problems which pertain wholly to activities foreign to their experiences. Lack of interest means a consequent carelessness in reading habits, an inaccurate interpretation, and an incorrect solution.

When the textbook is the source of the problems, the teacher's first duty is to see that each problem is made concrete to her pupils and that the words of the problem call up in the pupils' minds the correct ideas and images. Often the best way to make a problem or a group of problems real is to dramatize the situation involved. Many lists of problems in our newer arithmetics lend themselves readily to this idea. Although it is necessary to resort to dramatization more frequently in the lower grades, it should also be used in the upper grades with problems based on unfamiliar situations.

The problems in the older textbooks were usually organized according to the processes involved, with a list of problems on addition, another on multiplication, and so on. In most of the newer books, however, the problems are organized in a different way. The unity in a given list of problems, instead of being arithmetical, is social, the processes are varied but the problems are all based on the same life situation. Thus, we often find lists of problems on earning money, paying household expenses, buying

* In the Bibliography at the end of this chapter are listed, with annotations, the complete titles of the studies discussed throughout this chapter.

school supplies, sewing, and the like. Such an organization has two decided advantages. In the first place, the problems met in life are not all addition problems one day and multiplication the next; instead, on any one day they may be quite varied as to the processes involved, and yet related in that they all arise from one social activity. In the second place, if many problems have the same social setting, a greater opportunity is given to concentrate on this social situation and even to dramatize it if necessary.

Real Life Situations as Source. The only way interest in problems can be maintained is to originate or supply a large number and variety of problems arising from the needs of children. Pupils and teachers can find ample material in the activities about the school, the home, the church, and the community for the construction of problems from real life situations. Such work is excellent practice for pupils and often is the best means of enabling them to learn the content of problems and the variety of forms in which problem-ideas may be clothed.

No attempt should be made to reduce the children's statements to bare facts by culling out all descriptive and illuminating words. One of the chief reasons for the poor reading habits usually found in the arithmetic class is the fact that most problems are the bare statements of dead, uninteresting material.

Since counting, weighing, and measuring are outgrowths of the fundamental human needs, those concerned with food, clothing, and shelter, we should find the "Larger Home Idea" given to us by Mr. Betz an adequate source for the selection of data for "real problem situations." This "Larger Home Idea" involves the quantitative side of life found in (1) the home, (2) the school, (3) the state, (4) the nation, and (5) the world.

Monroe and Clark have compiled a bulletin on *The Teacher's Responsibility for Devising Learning Exercises in Arithmetic.* They have stated that the extent of the teacher's responsibility for devising problems becomes apparent when we recall that although children and textbooks provide a large number of problems, there are many gaps in the types of problems thus contributed. These gaps must be filled by additional problems contributed by the teacher.

Problem-solving should be based upon large thinking situations and the practical worth of the solving will be considerably greater if many problems relating to the same situation are solved in succession. Miscellaneous groups of problems are satisfactory for general

drill purposes, but problems solved in their natural sequence yield better returns both in fixing in the mind the principles of arithmetic which relate to a particular activity, and in acquainting the pupils with a body of systematic knowledge of social, economic, and industrial conditions.

To state the situations from which problems may be obtained entails an enumeration of almost all the vital activities of life, for problem-solving in arithmetic correlates with nearly all other subjects and experiences. This is amply proved by the partial list which follows:

1. Measurement of materials used in houses, stores, boxes, etc., heights of children, objects in room, distances jumped, materials for kite making, amount of water, sand, sawdust, etc.
2. Recognition of U. S. money through making change and buying milk tickets, penny lunches, tickets for picnic, articles at stores, car tickets, tickets for school entertainment, etc.
3. Cost of school equipment.
4. Class expenditures (class pins, class colors, etc.)
5. Playing store:
   a. Buying and selling.
   b. Cafeteria.
   c. Selling candy, popcorn, lemonade, etc., for class or school purposes.
6. School collections—Memorial Day, etc.
7. Class trips, school picnic.
8. Family trips by boat, auto, trolley, train.
9. Reading timetables.
10. Buying seeds, bulbs, etc.
11. Planning a small garden.
12. Planning for a party.
13. Making menus.
14. Following recipes.
15. Buying from a catalogue.
16. Amount saved when buying at a bargain or a special sale.
17. Computing the cost of several articles.
18. Purchasing Christmas presents.
19. Measuring gains in weight.
20. Telling time.
21. Earning money.
22. Selling magazines.
23. Helping in a store.
24. Sending parcel post packages.
25. Finding the cost of sending telegrams, express packages and freight packages.
26. Checking up laundry lists.
27. Keeping weekly personal accounts.

28. Sending money orders.
29. Writing receipts.
30. Keeping school records.
31. Buying baseball outfit.
32. Keeping milk account for school.
33. Filling a Christmas basket.
34. Manual arts work for boys, such as—
    a. Figuring the amount of lumber necessary for some project.
    b. Reading scale drawings.
35. Manual arts work for girls, such as—
    a. Figuring goods for a slip.
    b. Using an individual recipe for 10 persons.
    c. Changing a cake recipe which will provide 8 servings to one which will provide 12 servings.
36. Figuring the results of a spelling test.
37. Furthering a school-benefit play night.
38. Buying lunch equipment.
39. Guessing heights and verifying them.
40. Keeping class records and attendance by means of graphs.
41. Checking the family food supply.
42. Caring for pets.
43. Collecting bus fares.
44. Helping mother with shopping.
45. Keeping a home-problem notebook.
46. Keeping scores.
47. Keeping a temperature chart.

There must be opportunity for much care and skill in choosing, arranging, and using problems so that interest will aid learning. It is easy to go too far. It would be folly to try to select problem-work from a mixture of gymnastics, games, and childish whims. We should not expect to find an entire class of sixth grade children animated by a strong ruling purpose which involves a knowledge of decimal fractions. Many of the problems used in our fourth, fifth, and sixth grades will be taken from such life conditions as the following:

1. Various occupations.
2. Various industries.
3. Recreational activities.
4. Elementary science.
5. Statistics and graphs.
6. Civic and social activities.
7. Commerce.
8. Geography.
9. History.
10. Home economy.
11. Personal earnings.
12. School and class activities.

The wider the range of sources from which the various problems are taken, the more sure are they to approximate the needs of everyday life.

II. Essentials of Good Problems

Aim of the Problem. The great advances made in the teaching of arithmetic during recent years have come very largely through the broader conception of the purpose of the ideal problem. A problem is no longer considered merely a tool to train the mind.

In the elementary grades, children get several by-products from their study of a problem, such as the social insight into certain phases of life, the habit of looking upon quantitative relationships, and an appreciation of the value and power of arithmetic in doing the world's work. The problem is also used to apply, clarify, or rationalize arithmetical processes and facts and to motivate the drill work of these various grades.

Real Life Problems. The children's present interests are the fundamental factors to be considered in making or selecting problems. The early problems may be "story problems" about people or things present, and then about things owned by the children or familiar to them. The problems should have a content that the children help to select at that grade level. In the early years, interest and effort are determined much more by the wording of the problem than by the numbers and processes involved.

Wording of the Problem. If we wish to develop power, the wording of the problem must be so varied that the solution has to depend upon a rational analysis of conditions and not upon a mere act of memory. The ability to meet correctly any kind of simple arithmetical situation will be developed by meeting many and varied types of problems and by thinking through each type. This variety develops power and independent thinking on the part of the children and enables them to apply arithmetic to new situations.

It is now generally recognized that, above all, skill in problem-solving depends upon a comprehension of the facts of the case, facts which must be within the experiences of the children. This is the only way in which a problem can be made real and concrete to them.

The terms "real" and "concrete" have been interpreted in many ways. With some people, real has meant material, and the problems have been presented in terms of objects or words connoting very vivid images. Others have defined this quality in terms of use in the larger social world. Because certain problems actually occur

at the grocer's, the banker's, or the wholesaler's is no reason why they should be concrete to the children.

Opportunity for Dramatization. There is another social world, however, nearer to the child, which offers a more vital background. There is the opportunity to use the children's lives in their quantitative aspects by taking their plays, games, and occupations and introducing these situations into the teaching of mathematics. As the children's world expands from year to year, they will be carried by degrees from personal and local situations to those of general interest. Both the children and the teacher can provide this progression without devitalizing the facts and principles presented, because real situations create interest and thought opportunities, and, at the same time, lead to profitable end-results.

The conditions of classroom teaching place a limitation on this work in that the facts of the problems cannot always be presented concretely. They must often be described in words. The problems of life are often questions about situations or facts actually existing before the pupils' eyes; they are less often questions which the pupils put to themselves in connection with their past affairs or future plans; and they are least often questions put to them in words by another.

To sum up—a good problem is one that is based on the significant aims or purposes that are ruling factors in the children's lives at the time of learning. It provides for real situations or projects where reality is feasible. It is free from vocabulary difficulties. It offers opportunities for dramatization. It is neither much harder nor much easier than the corresponding situation outside of the school.

III. Classification of Problems

Lack of Uniformity. When we examine the problem lists in current arithmetics, we find a conspicuous lack of uniformity in the captions by which these lists are designated in different texts. Formerly, some of the problems given in arithmetics were listed under such captions as: Reduction, Measure of Capacity, Time and Wages, Linear Measure, and Percentage.

Changes in business practices and in the activities of life apart from the carrying on of business have created new arithmetical applications. Many of the titles formerly used as captions for problem lists have been discarded and new ones substituted. The

result is that at the present time we have no generally recognized plan for classifying the problems in arithmetic.

The buying and selling of commodities, and the like, create many arithmetical problems. This suggests that the sources of problems might be used as a basis for their classification; but an examination of the problems in our arithmetics reveals a number of problems whose source is not easily identified. In some cases the problem does not appear to be connected with any particular activity, and in other cases the suggested activity might be changed without affecting the problem.

Monroe and Clark. Monroe and Clark have listed two general classes of problems: (1) *Operation Problems* and (2) *Activity Problems.* Under Operation Problems are placed all those problems which are identified neither with a particular activity nor with an activity which has a special technical terminology. Under Activity Problems are placed all those problems which are identified with a definite activity of children or adults and which introduce a technical terminology. An elaborate study of the problems provided by texts resulted in the identification of 52 problem types in the field of "operation problems" and 281 problem types under "activity problems." The writers were compelled to acknowledge that this list of 333 problem types could not be regarded as final.

McNair. McNair and others give a workable classification of problems. The following statements are based very largely on the work of Mr. McNair.

1. *Simple and Complex Problems.* A simple problem is one involving but one operation. A complex problem is one involving more than one operation. To solve complex problems it is essential to separate them into simple problems.

2. *Concrete and Abstract Problems.* In arithmetic, a concrete problem is one which is based on the actual experiences of the pupils; whereas a problem is abstract when it in no way touches their experiences. Obviously problems which are concrete to some pupils are abstract to others. Too often a concrete problem is regarded as one in which the numbers used are definitely named. The fact that a problem contains "concrete" numbers does not make it a concrete problem. Problems given to grade children should be concrete, since such problems not only augment the interest but are solved with greater ease and with a fuller satisfaction.

3. *Oral and Written Problems.* An oral problem is one solved

without recourse to a pencil, while a written problem necessitates the use of a pencil. It is coming to be the opinion of many efficient teachers that fully half of the problems given should be oral. If the sole purpose of the problem is to impress the thought operation, then it is a saving of time to make the problem oral, since at least three oral problems can be solved in the time required for one that is written. Not only are oral problems valuable in establishing right habits of thought in arithmetic work, but they serve as an easy means of preparing the mind for the new presentation.

4. *Real Situation Problems.* Real situation problems are those which present to the children actual living experiences. Real situation problems are serviceable in that they make it easier for pupils to comprehend certain thought forms which must be taught.

5. *Original Problems.* These are problems "made up" by the children. Sometimes the data are given, but generally the children are asked to supply their own data. Original problems in the main have four uses. They cultivate the powers of observation, aid in modes of expression, give exercise in impressing correct thought forms, and encourage originality.

6. *Household Problems.* Household problems are most practical, since they deal with the daily living of all. Children should be required to originate problems of this type, using actual home experiences as data.

7. *Problems of General Interest.* These problems deal with money, World Series games, civics, population, etc.

8. *Occupational or Industrial Problems.* A moderate use of the industrial problems gives to pupils certain interesting information which should be possessed by educated people generally. The industrial problem enables the teacher to project arithmetic into the actual business interests of any community which supports an industry. One danger to avoid is the giving of a long and monotonous list of problems based on an industry which is foreign to the environment of the children. A second danger rests in giving data which are far-fetched or not true to the industry. We should bear in mind, however, that the fundamental purpose of the problem is to teach arithmetic, not the facts of some favored industry.

9. *Practice or Isolated Problems.* These problems are always accessible, since a large part of most texts is devoted to this type. As a means of training to right thinking, however, the practice problem is not as serviceable as it would seem to be, because the

necessary operation is too often suggested by some key word incorporated in the problem.

10. *Group Problems.* Group problems are those which pertain to the same subject or group of subjects. A list of twelve problems dealing with the shoe industry of Rochester is an illustration of this type of problem.

11. *Serial or "String" Problems.* When group problems are so arranged that they demand not only separate answers but a "grand total" answer, they become a problem series. The serial problems should reveal accurately the business transactions involved. The one vital objection to them is that one error early in the computation vitiates all the work that follows.

12. *Check Problems.* A check problem is one used to prove the correctness of the answer of a problem previously given. The check problem contains the answer of the original problem together with a part of the data. These problems serve a purpose in the cultivation of accuracy.

13. *Problems without Numbers.* These are numerical situations put in problem form, but not stated in terms of definite numbers. In teaching the solution of problems one may be more concerned with clear visualization, accurate comprehension of the conditions, and thoughtful development of sequential thought than with the actual manipulation and the answer. In such a case it is advisable to give problems that contain no numbers, and to require children merely to tell the processes which are to be followed to obtain the answer.

14. *Incomplete Problems.* Children may either complete an incomplete mathematical situation or formulate a problem to illustrate a given condition. The completion of problems, and the invention of new ones, may become a valuable class exercise. The children usually analyze a problem that is given them; now they must synthesize elements and relations to bring about a new situation. Invention of problems gives a new view of number relations, which leads to a greater comprehension, a deeper grasp, a broader application, and a firmer mastery of the various types.

15. *Project Problems.* There are four ways in which projects may be utilized in the classroom: (1) As introductory projects that are used for motivation or as a setting for the various arithmetical activities. (2) As cumulative projects that are used for reasoning or drill. (3) As review projects that are used for checks. (4) As

supplementary projects that are carried on for application or for purposeful enrichment.

IV. Emphasis on Problem-Solving

Difficulties in Problem-Solving. Every teacher of mathematics has at some time been confronted with the fact that pupils have great difficulty in problem-solving in arithmetic, and that they achieve more unsatisfactory results in this connection than in example-solving. The truth of this statement has been substantiated by several scientific investigators.

Considered from the point of view of the importance which authorities on arithmetic attach to problem-solving, the natural inference is that considerable attention should be given to this topic in textbooks on arithmetic, books on methods of teaching the subject, magazine articles, and courses of study. This, however, does not appear to be the case.

Newcomb. Newcomb made a study, the purpose of which was to determine the relative amount of space devoted to problem-solving in the several sources where such material should be found. The data for this study were secured through the examination of textbooks on methods, courses of study, teachers' magazines, school bulletins, textbooks in arithmetic. The results of the investigation show that the general average of the percentage of space from all sources devoted to problem-solving is 4.97 per cent of the total space devoted to arithmetic.

Eighth grade textbooks in arithmetic were found to vary from 1 to 9.8 per cent in the total amount of space given to the problem. Seventh grade texts were found to be somewhat more liberal: the amount of space given to the problem in these texts varied from 7 to 18.5 per cent.

The results of this study reveal a situation which is incompatible with the expressed opinions of almost all the authors investigated.

Spaulding. Spaulding made an analysis of six selected third grade textbooks with a view to determining the percentage of problems found. His conclusions are encouraging:

1. There is evident a tendency toward greater emphasis on problem-solving in contrast to the simple doing of examples of the earlier textbooks.

2. Makers of textbooks are coming to appreciate the need for making their problems representative of the fields of activity in which pupils are likely to be engaged.

It seems reasonable to conclude from the study of Newcomb that textbook problems may be a source of real want or waste in the teaching of arithmetic. May it not be that the very meager space given to instruction in problem-solving in the various appropriate sources has a silent influence upon teachers, thereby causing them to minimize this most important phase of arithmetic? Should there not be an amount of space devoted to problems in the various sources commensurate with the recognized importance of the subject?

V. Degree of Difficulty

How Difficult Should the Problem Be? The more one investigates and analyzes the various processes used in problem-solving in actual life situations, the more one is convinced that these situations are simpler than situations represented in the old type textbook and in some of the modern books.

To what degree of difficulty should the useful processes be carried? If this question can be answered correctly, it will enable teachers to use judgment and will guide them in the size of numbers or the number of places that should be used in the problems they offer. If they come as they should, from actual experience, the difficulty will take care of itself. When the teacher originates problems or dictates problems from books, a knowledge of the degree of difficulty is essential.

Wilson. Wilson has tabulated and graphed the actual distribution of 4,416 addition problems used in four selected occupations as to the number of places involved in the largest addend. It is evident from these graphs that the numbers added are relatively small. He reported that there are less than two per cent of addition problems with addends of five places or above. Almost all the problems have either two-place or three-place addends. In handling money a two-place problem means one in which the amount is less than one dollar.

In subtraction the minuend which is most common is the three-place minuend. Because subtraction is most frequently used in making change, this means that the most common amount given in payment for which change must be made is the one-dollar, two-dollar, or five-dollar bill. The next most common subtraction situation involves two places, followed in order by four, one, five, and six places.

The multiplication problems are classified on the basis of the number of places in the multiplier. The one-place multiplier makes up 53 per cent of all multiplication. The farmers and the merchants deal more often with problems that involve two-place multipliers.

The table on division shows that the two-place divisor is most common, although it is nearly equaled in frequency by the one-place divisor. These tables and graphs reveal the simplicity of the work in the four fundamental processes in actual life outside of the school.

The degree of difficulty is often dependent upon other elements than the size of the number involved, the number of processes, or the difficulty of these processes. When a child meets a new situation, he is often confused, not because of the mathematics of the problem, but because he does not understand the situation, the vocabulary, the phraseology, or the form of the problem.

Hammond. Miss Lucy Hammond, critic teacher of Grade 5BI, Rochester Normal School, made a study of the difficulty in the wording and form of problems. This study showed the great variety of ways in which the same quantitative facts may be stated and arranged in problems, and the degree of difficulty encountered in the various statements. The following is an illustration of this study of problem difficulty:

Problem situation: "John has $12.75 in the bank. He worked 8 hours at $.30 an hour and added this to his savings. How much money did he have then in his bank?"

1. If John has already saved $12.75 and then adds to this what he earns, working for 8 hours at $.30 per hour, what has he saved altogether?

2. John worked at the rate of $.30 an hour for 8 hours and put this amount with his savings of $12.75 in the bank. Find his total savings.

3. John wished to add to his bank account which amounted to $12.75. For each hour he worked on Saturday he received $.30. If he worked 8 hours, what was the amount on his book?

4. When John worked 8 hours and was paid $.30 for each hour, he added this to his bank account of $12.75. What was his total then?

5. John earned $.30 an hour for work which took 8 hours. He had saved $12.75 and he added to this bank account the money he had earned. How much did he deposit in all?

6. John had put away $12.75 in the bank. Later he worked at a newsstand and earned $.30 an hour. How much were his total savings, if he worked 8 hours?

7. Find the total amount in John's bank book, if the first deposit was $12.75 and the second deposit was his earnings for 8 hours at $.30 an hour.

Thorndike. Thorndike emphasizes the following sources of difficulties which arise in framing problems, and warns teachers to avoid them:

1. Rare and unimportant words which occur in the first fifty pages of some well known lower grade arithmetics.

absentees	camphor	hesitation	mentally	purchased
admitted	cinnamon	income	mercury	respectively
alternate	confectioner	installment	phaeton	supply
baking powder	deposited	insurance	proprietor	treasury

2. Misleading facts and procedures.

"At $.13 a dozen, how many dozen bananas can you buy for $3.12?"

3. Trivialities and absurdities.

"From the Declaration of Independence to the World's Fair in Chicago was 9 times as many years as there are stripes in the flag. How many years was it?"

4. Useless methods.

"If I set 96 trees in rows, sixteen trees in a row, how many rows will I have?" This method forms the habit of treating by division a problem that in a real situation would be solved by counting the rows.

5. Problems whose answers would, in real life, be already known.

"A clerk in an office addressed letters according to a given list. After she had addressed 2500, ⅜ of the names on the list had not been used; how many names were in the entire list?"

6. Needless linguistic difficulties.

"If a croquet-player drove a ball through 2 arches at each stroke, through how many arches will he drive it by 3 strokes?"

7. Ambiguities and falsities.

"How many lines must you make to draw ten triangles and five squares?" (It can be done with 8 lines, though the answer the book requires is 50.)

"John earned $4.35 in a week, and Henry earned $1.93. They put their money together and bought a gun. What did it cost?" (Perhaps $5, perhaps $10. Did they pay for the whole of it? Did they use all their earnings, or less, or more?)

The distinction between life situation problems and verbal problems should be considered. Strictly speaking, the life problem is the situation itself and not the word description of the situation. The reason why the child is confused by the verbal problem is that he has difficulty in interpreting and passing from the verbal description to the problem itself. If a child goes to a store and buys a tablet and a pencil, he is not confronted by the interpretation of the situation. He knows that the cost of the articles will be added and that the sum will be subtracted from the amount he gives the clerk. When this same problem is given to him by the teacher or is found in the book, his reactions are not the same. He must then be

able to read and understand the problem, to visualize the situation, to become a part of the picture, and to determine the processes needed to solve the problem. It is possible for a child to be able to deal successfully with life situation problems and still fail hopelessly in solving verbal problems.

VI. Process of Problem-Solving

Selecting the Process. Although pupils have difficulty with the various steps in the solution of problems, it is the selection of the process that is usually the most difficult step and gives the greatest trouble. Pupils will always have trouble in planning the solution of problems unless the process is connected in their minds with the concrete situation in which it is to be used.

Teachers realize that it is their function to teach their pupils to add, subtract, multiply, and divide, but they do not always realize that it is just as important to see to it that the pupils become familiar with the various types of concrete situations that involve these processes. If a connection is to be formed in the pupils' minds between the processes and the concrete situations calling for these processes, the two must be presented together. This means that when a new process is taught, concrete problems that involve the process must be given, and from that time on the abstract drill on the process and the use of the applied problems must both be emphasized. By using these processes in concrete problems the pupils gradually come to recognize, for example, subtraction situations as such and to distinguish them from addition, multiplication, and division situations.

Less abstract drill is necessary when the processes are used in applied problems and when the abstract and applied phases are kept as closely connected as possible. Later, the pupils should be helped to generalize their ideas of addition, subtraction, multiplication, and division situations. One of the best ways of doing this is to make use of generalized problems, that is, problems without numbers, which ask for the processes only.

To develop a systematic method of attack in solving all problems in various grades, the pupils must go through simple types of one-step, two-step, and three-step problems. Problems should be stated in various ways and should be descriptive of a wide variety of material within the range of the pupils' own experiences and interests. Two-step problems should not be given until the pupils have

displayed a rather complete mastery of one-step problems; three-step problems should be deferred until there is a similar mastery of the two-step problems.

Suggestions. The following suggestions may be helpful in the classroom:

1. Have the pupils make up problems, after giving them specific directions, such as,
   (a) Make up a problem in which you multiply.
   (b) Make up a problem in which you add and subtract.
2. Give the pupils problems and have them indicate the processes by writing the letters suggesting the operations, as SA, MA, and the like.
3. Give the pupils problems on slips of paper and have them make up similar problems from their own experiences.
4. From lists of review problems in textbooks or on slips of paper, have pupils select and list by number the problems in which the same process or processes must be used.
5. Originate problems without numbers for which the pupils will give the processes.
6. Originate or collect problems which cover the various processes and see that pupils are able to meet any of these situations by solving any such problem.

Teachers should avoid using too many problems in any particular process as application or review work. Such work has little or no value as a reasoning exercise. Unless problems require a variety of responses, so that the children must think in every case, they are valuable merely as an exercise in mechanical computation.

The success of the pupils in problem-solving depends very largely upon the strength and permanence of the bonds formed in the minds of the children between a specific, concrete situation and the related arithmetical process. The forming of these bonds is the all-important work of the elementary school.

VII. Causes of Failures in Problem-Solving

Literature on the Subject. Problems are a constant source of trouble for pupils and teachers. The only remedy is directed practice in problem-solving. In recent years an extensive literature has arisen which deals with the various causes of failure in problem-solving.

Second Yearbook. The *Second Yearbook of the National Council of Teachers of Mathematics* has listed the following causes of failure in problem-solving:

1. The language used in the problems is too difficult. It is beyond the reading standard for the grades in which it occurs.
2. The pupils have not had sufficient training in interpreting thought from silent reading.
3. The pupils lack understanding of the technical terms involved.
4. The situations described by the problems are not understood by the pupils, because they are outside the range of the pupils' experiences.
5. The fundamental combinations, facts, or processes called for in the solution of the problems have not been habituated.
6. The pupils are unable to see the relations between the steps called for in the solution of the problem.
7. The pupils are so burdened with undue labeling and elaborate indication of steps that their minds are diverted from the real process of solution.

Fourth Yearbook. The *Fourth Yearbook of the Department of Superintendence* has listed:

1. Lack of general ability in silent reading.
2. Lack of familiarity with technical terms in arithmetic.
3. Carelessness in reading.
4. Lack of experiences necessary to understand the setting of the problem.
5. Inadequate skill in computation.
6. Lack of knowledge of such essential facts as tables of weights and measures.
7. Inability to see the relationships in the problems so as to choose the proper operation.
8. Inability to do reflective thinking.

Stevenson. Stevenson has listed six principal causes of failure in problem-solving:

1. Physical defects.
2. Lack of mentality.
3. Lack of skill in fundamentals.
4. Inability to read, which of necessity affects the ability to read arithmetic problems.
5. Lack of general and technical vocabulary.
6. Lack of proper methods or technique for attacking problems.

Osburn. Osburn has made a study of the reading difficulties encountered in problems. He listed nine causes of misunderstanding:

1. Lack of vocabulary.
2. Failure to read or see all the elements in the problem.
3. Failure to resist the disturbance caused by preconceived ideas.

4. Inability to read between the lines.
5. Failure to understand fundamental relations, particularly those of the inverse type.
6. Failure to make a quick change of mental set.
7. Failure to generalize or transfer meanings.
8. Failure to interpret cues correctly.
9. Response to irrelevant elements.

Lessenger. Lessenger has made a study of the effect of difficulties in reading, as related to problem-solving. He found that in a mixed-fundamentals test many children made errors because they did not read accurately the directions indicating the nature of the processes. By allowing full credit for all examples solved correctly, regardless of whether the operation actually used was the one called for in the printed instructions, he found that the loss in score due to faulty reading amounted to from 6.1 to 10.1 months of arithmetic age. When special training in reading arithmetic problems was given, the amount of error due to this cause was greatly reduced. In the case of one group of sixty-seven children the gain in score due to improvement in reading amounted to 9.1 months of arithmetic age.

Morton. Morton has written a series of articles dealing with the solution of arithmetic problems. The series concludes with a number of case studies in which extensive descriptions are given of the processes used by individual pupils. One interesting result of the study is the high correlation shown to exist between ability in the fundamental operations and in problem-solving. It was found that the scores on the arithmetic-fundamentals test correlated more highly with the problem scores than did any other measure, *except that secured by the verbal-intelligence test.*

Bradford. Bradford has reported a study in which a group of children was given a series of arithmetic problems impossible of solution. The purpose of the experiment was to determine how many of the children would show genuine critical thought in discovering that the problems could not be solved, and how many of them would simply go through the formal manipulations without recognizing that the problems were impossible. The following problem is typical of those which were employed.

"How fast is a cloud moving across the sky, if from my bed I watch it cross a window one yard wide, in one minute? Give the answer in miles per hour."

On the basis of his experiment, Bradford concluded that many right answers are obtained under ordinary classroom conditions not as the result of genuine critical thought, but as the result of suggestion. When the pupils were told that some of the problems were impossible of solution and were required to make critical reactions to them, the number of correct answers showed a substantial decrease. The experiment is of particular interest in view of the improvement in critical thinking which is commonly supposed to result from solving problems in arithmetic.

Buckingham. Buckingham's brief report on evaluating arithmetic problems has shown great keenness in the choice of problems adapted to the median ability of a grade. It is noted that children do better work upon problems formulated by the teacher than upon problems found in textbooks.

Committee of Seven. For two years the Committee of Seven of the Superintendents' and Principals' Association of Northern Illinois, with the coöperation of teachers and pupils in a number of schools, attacked the question of problem-solving in a number of ways. The results of the experiments appear to shed new light on the subject and to offer practical suggestions which will be generally useful. Tests were given to children in Grades 3, 4, 5, 6, and 7. The number of children tested ranged from three hundred to more than a thousand. In this chapter the experiments will be briefly described and the general outcome of each given.

The Committee, working under the guidance of Superintendent C. W. Washburne of Winnetka, Illinois, attempted to answer certain critical questions which are listed below.

1. *Is inability to visualize the situation dealt with in the problem an important cause of failure to solve the problem?*

Washburne and his committee attempted to answer this question only for comparatively simple one-step problems. The procedure consisted of giving the children such questions as this: "Mary wishes to buy a gift for her sister. She went to the toy store to buy a doll." Then followed four pictures. In one, Mary was standing at the counter of a fruit store; in another, she was in a drygoods store; in a third, in a candy store; and in the fourth, in a toy store. The children were to underline the picture that told about the problem. This type of question was repeated in a variety of forms and in two different series of tests.

The results showed that practically all the children answered the

questions correctly. Apparently, the children did not encounter in simple one-step problems the reading difficulties which are commonly supposed to hamper pupils in visualizing the situations involved.

2. *Is there any relation between ability to solve problems and ability to make some formal analysis?*

A great deal of time and energy was spent on this question. Tests were prepared, given, analyzed, revised, re-given, and re-analyzed repeatedly. One-step problems, two-step problems, problems involving various combinations of processes, and problems suited to the earlier grades and to the later grades were tried. The results were analyzed in various ways. They were directly contrary to what was expected, so the papers were checked and rechecked.

An analysis of all the tests in all the grades showed that in every case there was little or no relation between ability to solve the problem and ability to take any of the other steps. The children analyzed a problem correctly and solved it incorrectly or solved it correctly and analyzed it incorrectly almost as often as they both solved and analyzed it either correctly or incorrectly.

Washburne was forced to the conclusion that ability to make the type of formal analysis frequently taught in school has practically no relation to ability to solve problems.

3. *To what extent is unfamiliarity with the situation involved in the problem, or with the materials with which the problem deals, a cause of failure to solve the problem correctly?*

This question was studied in the following manner. Tests were devised with pairs of problems of the same mathematical difficulty, one of each pair dealing with a less familiar situation or with less familiar materials than the other. The following is an example of these series of tests:

Three bars of chocolate sell for 10 cents. How many bars can I buy for 40 cents?

In France two liters of petrol cost 9 francs. How many liters of petrol can be bought for 90 francs?

These tests were given in two different schools. The score on problems involving more familiar situations was 14 per cent higher in School No. I and 12 per cent higher in School No. II than the score on problems involving less familiar situations. These results indicate that while the element of unfamiliarity with the situation enters in as a difficulty in problem-solving, it is not so large an

element as might be supposed. Furthermore, well chosen textbook problems for class work contain very few such unfamiliar situations as are presented in some of the problems in the tests.

4. *Do pupils who can manipulate numbers correctly in the processes involved and who can solve problems when the numerical work is simple enough to be done mentally fail to carry over these abilities in working similar problems that require pencil and paper?*

As an example, if a child can divide 132 by 3 on paper and can solve mentally a short-division problem involving the use of very easy numbers, can that child solve a short-division problem involving the use of the written process? To answer this question a test was devised which was made up of three sets of problems, each set involving (1) the written process alone, (2) a problem simple enough to be done mentally, and (3) a similar problem with numbers large enough to require the written process. The sets covered subtraction, multiplication, and division and were given in a mixed order.

The results indicate that pupils' inability to carry over a process they know to a problem requiring written work is a frequent cause of difficulty, even when they can do a similar problem involving small numbers. This difficulty is most evident in the lower grades where the processes have just been learned, but it tends to disappear as the processes become more fully automatic.

5. *Is lack of facility in the accurate use of arithmetic mechanics a common source of error in problem-solving?*

A study of the errors involved in the tests just described and in a similar test given in a different school shows that most of the errors are as likely to occur in the working of examples as in the solving of problems.

During the second year of its investigations, the Washburne committee confined its work to an intensive study of the relative merits of three methods of training children to solve problems. In this study the coöperation of eighteen different schools was enlisted, involving records from 763 children in Grades 6 and 7. The children in each of these grades were divided into two equal, parallel groups. That is, these groups were equal in their ability to solve problems and to deal with arithmetic fundamentals and the children were approximately equal on the basis of mental age, chronological age, and the judgment of the teacher.

Each school was expected to test the relative merit of two of the following methods:

Method I. To train children in the solving of problems by giving them large numbers of problems—no special technique.

Method II. To train children to analyze problems—a definite technique of attacking each problem.

Method III. To train children to see the analogy or similarity between difficult written problems and corresponding easy oral problems and thereby to decide what process to use in attacking the difficult problems.

Form I of the problem-solving test was given all children, and at the end of six weeks of training they were retested with Form II of the test.

The conclusions drawn were that in all cases the children made remarkable gains. This seems to indicate clearly that concentrated attention, even for a few weeks, on solving problems by *any* method brings a rich reward. Training in the seeing of analogies appears to be equal or slightly superior to training in formal analysis for the superior half of the children; analysis appears to be decidedly superior to analogy for the lower half; but merely giving many problems, without any special technique of analysis or the seeing of analogies, appears to be decidedly the most effective method.

In general, then, these investigations seem to show that, on the whole, the children who were taught no special technique of solving problems, but simply solved many problems, surpassed those who spent time learning a special method of solution. Dr. Washburne states, however, that the problems used were very simple and required little analysis. He made no use of problems that involve a number of steps for solution, although this type primarily would check the value of analysis. Hence the conclusions of the Washburne committee would require further testing in the case of complex problems.

Newcomb. Mr. Newcomb of Ada, Oklahoma, also conducted an interesting experiment to ascertain the value of analysis in problem-solving. The test problems selected required considerable analysis. He also made use of experimental and control groups. A definite technique was taught in each experimental class and each pupil was required to use this technique in a prescribed manner, as follows:

1. Read the problem carefully and thoughtfully.
2. State what is given.
3. State what is to be found.

4. Write the processes you will use.
5. Write the approximate answer.
6. Solve the problem in the space below.
7. Check. Does your answer seem reasonable?

The results showed that an average gain of 5.1 per cent in speed and 2.9 per cent in accuracy was made by the control classes, while an average gain of 22.8 per cent in speed and 5.5 per cent in accuracy was made by the experimental classes. The result secured is convincing enough to commend the method for trial by teachers who are experiencing difficulties in teaching pupils how to solve problems.

Banting. G. O. Banting made a profitable study of "The Difficulties in Reasoning in Arithmetic," in the Waukesha schools. An explanation and a summary of this study are given in the *Denver Arithmetic Course of Study,* 1926, or in the Bulletin of the Department of Elementary School Principals, *The Second Yearbook,* July, 1923.

Part Two

Introduction. It is the purpose of this section to offer a constructive policy for the improvement of problem-solving, through the tabulation of some recent investigations and research studies, the formulation of major objectives, the elucidation of the psychology of problem-solving, and the listing of standard tests.

I. Recent Investigations and Experiments in Problem-Solving

Types of Investigation. Three distinct lines of investigation should be mentioned. First, inquiries as to the various kinds of arithmetical problems which ought to be included in a course. Second, investigations to determine the types of problems which pupils will encounter in school and in adult life. Third, inquiries dealing with the psychology of problem-solving.

The investigators of these questions are using all the methods of analysis that are available. They are employing tests to a large extent. These tests are used in part to criticize or to validate existing problems. They will also help toward the constructive revision of the scope and content of problems. Many more scientific inquiries will probably be required, however, before problems can be reconstructed in such a way as to meet fully the needs of all pupils.

A few of the recent outstanding investigations and experiments will be summarized very briefly.

Gard. Gard made a careful analysis of the reasoning processes of a small number of adults. These subjects solved seven problems and then gave full accounts of their mental processes. The actual mental processes were characterized by many deviations from the well known series of steps in the method of problem-solving. The experiment showed that previously established habits of procedure had marked influence on the method of attack. Familiarity with the types of problem resulted in greater speed. In some cases the reasoning seemed to be merely a guess or an appeal to the method of trial and error.

After noting the many deviations from typical processes of reasoning in the case of these adults, one begins to doubt the wisdom of too great an insistence on cumulative reasoning in the case of immature children. Apparently the reasoning necessary in many problems is a complex operation. If this be true, the teacher is not justified in assuming that all elementary school pupils can rise to the level of systematic reasoning required in complex problems.

Terry. Terry made an elaborate investigation of the reading of problems. His study shows that when reading numerals one divides the material being read into much smaller units than when reading of arithmetic problems. The textual part of an arithmetic numerals at a time. In reading 436287, an ordinary reader is likely to break up the material in some such way as this: 4, 36, 2, 87. The divisions made to facilitate recognition are not always the same. The same reader may later read the numerals by subdividing as follows: 43, 62, 87. The fact that the range of recognition for numerals is so different from the range of recognition for words and letters, where as many as twelve letters can be grasped at a single glance, explains why it is that pupils are often confused in their reading words. A reader very seldom apprehends more than two problem requires one type of attention, while the numbers call for an entirely different type. As a result, the child who is reading such a problem is stopped in his progress through the problem every time he comes to a number. To keep numbers in mind requires a type of memory entirely different from that which is required to recall the coherent words and phrases which make up the verbal part of the problem. The trained adult, when confronted by the two different intellectual demands described, adjusts himself to the difficulty by reading the problem twice. During the first reading he gets the story of the problem and discovers the arith-

metical operations which are necessary in order to reach a solution. During the second reading he notes the exact digits of which the numbers are composed.

Important lessons for the teacher are suggested by this analysis. In the first place, it becomes evident why pupils are often greatly confused in reading arithmetic problems. In the second place, it becomes equally evident that pupils must be given much opportunity to cultivate effective reading habits if they are to escape this confusion. They should be trained in the reading of the numerals and in the methods of interpreting the verbal part of the problem. This means attention to the vocabulary of the problem, careful explanation of many terms which are now too often taken for granted, and a study of the grouping of numbers. There is no doubt that the arithmetic reading habits of pupils involve many complexities which will require additional systematic study.

Wilson. Wilson published a report of work carried on in various centers in the effort to find out from business men and others what problems they actually encounter in everyday life. He concludes:

1. The problems solved in actual life are brief and simple. They chiefly require the more fundamental and the more easily mastered processes.
2. In actual usage, few problems of an abstract nature are encountered. The problems are concrete and relate to business situations. They require simple reasoning.
3. The study justifies consideration of this question: After the development of reasonable speed and accuracy in the fundamentals and the mastery of the simple and more useful processes, should not the arithmetic work be centered largely around those problems which furnish the basis for much business information?
4. May we not hope through the use of large informational problems and situations in the upper grades to secure a more intellectual application of arithmetic to actual life situations, that is, the use of more arithmetic in the productive work of the kitchen, in intelligent buying, in accounting, in profitable saving and banking, and the like?

A group of students in Boston University made a survey of the uses of arithmetic in Boston and vicinity. Each member of the class canvassed a number of pupils and obtained the problems in arithmetic encountered by their parents in the course of their ordinary experiences during a period of two weeks. The combined efforts of the class secured 5,463 problems. The data were classified in general according to the Wilson plan. Sixty-two different occupations were represented. It is interesting to note that 96.8 per cent of the 5,463 problems tabulated are general problems and that 3.2 per cent are classified as vocational problems.

Tables were prepared which arranged the problems with regard to the four fundamental processes. They show that a total of 81 per cent of all the problems involve the four fundamental processes and that multiplication is the most common process. This survey shows a large percentage of one-place multipliers and one-place divisors, while the minuends show a tendency to be large. A ten-dollar bill offered in payment means a four-place minuend in subtraction, while a five-dollar bill means only a three-place minuend.

One of these tables shows the distribution of 2,786 fractions. The fraction $\frac{1}{2}$ has a frequency of 90.955 per cent; $\frac{1}{4}$ stands next with a frequency of only 2.872 per cent. Other common fractions arranged according to frequency are as follows: $\frac{3}{4}$, $\frac{1}{3}$, $\frac{5}{8}$, $\frac{1}{8}$, $\frac{2}{3}$, $\frac{7}{8}$, $\frac{3}{8}$, $\frac{1}{6}$, $\frac{3}{16}$, $\frac{1}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{1}{10}$, and $\frac{3}{10}$. The denominate numbers do not occur except as simple measures—cents, dollars, yards, and quarts; there are no reductions and no compound number processes. In almost all cases the decimals relate to simple discounts.

Mr. Wilson claims that the general conclusions derived from this study merely confirm the conclusions drawn from similar studies.

Many educators have questioned the desirability of using social utility as an adequate basis for the selection of arithmetic material for any course of study, or for children's complete preparation in school. These social needs are very important, but educators do not wish to encourage surface learning through too narrow and too meager a selection of material. There is always the danger of gaps through the omission of fundamental principles, if one depends entirely on socialized activities.

Knight and Buswell. Knight and Buswell formulated some excellent suggestions, published in the *Second Yearbook of the National Council of Teachers of Mathematics,* concerning additional research studies in this field of investigation. To quote:

1. We should find out what types of problems children can do and should be required to do. We constantly over-estimate the child's ability to reason. Most problems given to children are too difficult for them.

2. We should discover the most effective classroom technique for teaching the skill of problem-solving in elementary school arithmetic.

3. We should attempt a standardization of the vocabularies used in the different books and a correlation, grade by grade, between the vocabularies used in arithmetics and the vocabularies encountered in general reading in the elementary schools.

4. There is need of a detailed individual diagnosis of the actual thinking carried on by children in solving the ordinary problems presented in arithmetic; perhaps, to begin with, those dealing with one particular unit only.

Until such a survey of children's reasoning is available, it will be difficult to supply suitable instructional material for problem-solving.

5. There is need of the preparation of a textbook which is composed essentially of explanatory material, with an accompanying manual of practice exercises. The space which has been previously used for examples and problems may well be given to detailed instructions to the pupils relating to methods of procedure, an explanation of arithmetical operations, and a presentation of social situations in which arithmetic is to be applied.

6. The following are possibilities for the development and application of remedial instruction units:

a. Exercises stressing vocabulary.
b. Exercises stressing problem comprehension.
c. Exercises stressing what is given in the problem.
d. Exercises stressing what is called for in the problem.
e. Exercises stressing the estimation of answers.
f. Exercises stressing choice of procedure.
g. Exercises stressing relationships in problems.

As a summary of these few investigations it may be stated that problem-solving is a complex type of mental work. For pupils of low-grade mentality, arithmetic is the most difficult of the school subjects. Problem-solving is one of the most serious sources of non-promotion. It is a form of experience which is not demanded in large degree by the children's untutored interests, and excellence in problem-solving is an infallible mark of general intelligence.

II. Objectives of Problem-Solving

Objectives. One prime objective is to train boys and girls to solve problems met in actual life situations. No matter what the pupils' lives are, they present situations that have a numerical trend, and it is the privilege of the school to prepare its pupils to meet these activities successfully. In order to do this, many life conditions are brought into the school and constitute the applied problems in our arithmetic.

Problems vitalize arithmetic, give purpose to it, and afford a medium for utilizing skill in computation. Good problems arouse keen interest and afford a quantitative interpretation of a variety of experiences.

Owing to the constant contact of individuals with number relationships in school and out, the school should set a high standard in the development of ability to interpret, comprehend, and solve problems that arise in the activities of the world. Hence these objectives should always be kept in mind:

1. A knowledge and automatic mastery of the fundamental principles and operations.

2. An ability to see numerical relationships and the power to formulate such relationships.

3. The acquisition of power to utilize the fundamental skills and concepts in the solving of problems.

Ruskin has said, "The difference between great and mean art lies not in definable methods of handling, or styles of representation, or choices of subjects, but wholly in the end to which the effort of the painter is addressed." His words can be applied to the teaching of problems as distinctly as to the art of which he wrote.

If we as teachers see the great things of life, we shall say with David Eugene Smith, "Our work is great in the classroom if we feel the nobility of that work, if we love the human souls with whom we live more than the division of fractions, if we like problem-solving so much that we make our pupils like it, and if we remember that our duty to the world is to help fix in the minds of our pupils the facts and principles of number that they must have in life."

III. Psychology of Problem-Solving

Recent Literature. The most recent books on the teaching of mathematics give some attention to psychological questions. Thus, the arguments used in the debates on the transfer of training are discussed more or less critically. There have been some recent experiments on this issue.

Poffenberger. The following evidence of transfer is adduced by Poffenberger:

1. Where there are no identical bonds between stimulus and response in the two processes, the influence of one process upon another will be neither positive nor negative, i.e., there will be neither transfer nor interference.

2. Where there are identical elements in the two situations or where a given process involves one or more bonds previously formed, there will be a positive or transfer effect.

Knight and Setzafandt. Knight and Setzafandt show in the following summary that training transfers:

Characteristic of the whole-or-none attitude of American thinking, when once our faith in general transfer was taken from us by competent research, notably the work of Thorndike, we went over, for some time, to the theory that there was *no* transfer whatever. We learn exactly what we practice and nothing more.

Absence of any transfer is as false and probably as harmful a notion to

use in problem-solving as is its antithesis, complete and magical transfer through the power of the mind. The problem of transfer at the present time is concerned not with the presence of transfer or its utter absence, but with such questions as:

1. In what types of skill is there enough transfer to use?
2. On what level of mastery does transfer in useful amounts begin to operate?
3. In what ways can transfer be facilitated and increased?
4. Are transferred skills or skills acquired in one set of data as strong, as lasting, and as easily operated in another set of data? How much is lost by transfer?

The present status of the transfer of learning seems to be: *Transfer exists to the extent that the same elements or skills are used in the new situation. Transfer is often small because we fail to realize that the same elements or skills could be used. If we wish transfer or the application of skills acquired in one situation to operate in another situation, we must train in the ability to look for uses of old skills. The old skills will not transfer by themselves except when conscious similarities are very close.*

Thorndike. Thorndike tends to emphasize the great variety of bonds involved in arithmetic and calls attention to the necessity of giving each bond separate emphasis. He says that *the mind works not only by association,* by connecting this situation with that response, *but also by dissociation* or analysis, by breaking up a total situation into its elements. The abstract and general notions of human thinking are mental products which come, not by putting things together only, but by separating them into parts.

The child at school to whom we wish to teach the abstract thing, number—for instance, the abstract quality of sevenness—is given seven apples, seven blocks, seven papers, seven sticks, seven chestnuts; is allowed to draw seven lines, to move his arm seven times, each time in association with the word *seven.* By having the *seven* quality constantly present, but in connection with all sorts of accessory qualities, he comes to feel the numerical aspect of the seven by itself as a separate elementary thought in his mind. This is *the law of dissociation.*

Judd. Judd adds that after this dissociation takes place and the child recognizes a relation and a common element, the child must apply this dissociated element to many new situations, for in so doing he generalizes the element. When the generalized element is used freely and perfectly without conscious effort, application is evident. The child has thus learned to *dissociate, generalize,* and *apply* common elements.

This same psychology should be applied to problem-solving. We should provide for various problems in which many common elements recur constantly, but in each case with different surroundings or context. The element or idea which is thus felt with many different associates or situations comes to fuse with none of them, wins an independent existence, and feels itself as an idea. The child thus abstracts and masters separately each phase of the problem for some time. These various parts should then be put together so that all the phases of the problem will be mastered collectively. Thus, in problem-solving analysis and synthesis, dissociation and association, breaking the problem into parts and putting the parts together, provide for emphasis and mastery of each ability individually and collectively.

IV. Standard Tests

A List. Following is a list of standard tests in arithmetic that can be purchased for testing pupils in problem-solving.

1. Buckingham Scale for Problems in Arithmetic. Buckingham, B. R. Public School Publishing Co., Bloomington, Ill., 1920. Price $.80 per 100.

   Grades and Forms: Division I for Grades 3 and 4; Division II for Grades 5 and 6; Division III for Grades 7 and 8. Forms 1 and 2 for each division.

2. Compass Diagnostic Tests in Arithmetic. Ruch, G. M., Knight, F. B., Greene, H. A., and Studebaker, J. W. Scott, Foresman and Co., Chicago, Ill., 1925. Price $5.00 per 100.

   Grades and Forms: Test 17 for Grades 5 and 6; Test 18 for Grades 7 and 8. One form only.

3. Monroe Standardized Reasoning Test in Arithmetic. Monroe, W. S. Public School Publishing Co., Bloomington, Ill., 1921. Price $.80 per 100.

   Grades and Forms: Test I for Grades 4 and 5; Test II for Grades 6 and 7; Test III for Grade 8. Forms 1 and 2 for each part.

4. Otis Arithmetic Reasoning Test. Otis, A. S. World Book Co., Yonkers, N. Y. Price $.40 for package of 25.

   Grades and Forms: Test for Grades 4 to 12. Forms A and B.

   This is Test 5 of the Otis Group Intelligence Test. It can be purchased separately.

5. Peet-Dearborn Progress Tests in Arithmetic. Peet, H. E., and Dearborn, W. F. Houghton Mifflin Co., Boston, 1920. Price $1.20 for package of 24.

   Grades and Forms: Series I for Grades 4 and 5; Series II for Grades 6, 7, and 8. One form only.

   The test consists of five parts—addition, subtraction, multiplication, division, and problems.

6. Spencer Diagnostic Arithmetic Tests. Spencer, P. L. Bureau of Administrative Research, University of Cincinnati, Cincinnati, Ohio. Price $2.00 per 100.

   Grades and Forms: Test III for Grades 7 and 8. One form only.

This test has two parts, one part in problem-analysis and the other part in problem-solving.

7. Stanford Achievement Test, Arithmetic Examination. Kelley, T. L., Ruch, G. M., and Terman, L. M. World Book Co., Yonkers, N. Y., 1922. Price $1.00 for 25.

   Grades and Forms: Test for Grades 2-8. Forms A and B.

This is part of the Stanford Achievement Test but it is also published separately. The test consists of two parts; one in computation and one in reasoning.

8. Stevenson Problem Analysis Test (Diagnostic). Stevenson, P. R. Public School Publishing Co., Bloomington, Ill., 1924. Price $1.00 per 100.

   Grades and Forms: Test I for Grades 4-6; Test II for Grades 7-9. Forms 1 and 2 for both parts.

9. McDade Cards. Cabinet Form. Plymouth Press, Educational Publishers, 7850-56 Lowe Ave., Chicago, Ill., 1927. $8.00 per cabinet.

Aids for individual child in problem-solving. Self-check. Timed without aid of teacher.

Part Three

Introductory. In the endeavor to formulate a definite, workable policy for individual and class improvement in problem-solving, the writer with the coöperation, the valuable suggestions, and the ready adjustments of two critic teachers, Miss Mildred Seekins and Miss Mary Caragher, set up the following experiments which were carried on and graphed at the City Normal School. It is hoped that in some small way the experiments may contribute to the improvement of method in problem-solving.

I. First Experiment at the City Normal School

Grade 6 B-2	Critic Teacher—Miss Caragher
Number of Pupils—25	March 29 to May 20, 1927.

Purpose. To determine the difficulties that pupils had in solving problems, to apply remedial instruction, and to increase, if possible, the ability to solve problems.

Procedure. Tests were given at the beginning of the experiment and at the end of the remedial instruction, to find individual disabilities and to measure the results of remedial instruction. The test used was the Compass Diagnostic Test XVII—Problem Analysis—Elementary. Form A was used for both tests.

After giving the first test, individual cards were made for each child to show his weaknesses and to simplify regrouping according to disability. The class was regrouped each day according to needs. Children not needing remedial instruction of any particular type spent their time doing written problems.

Remedial Instruction. For twenty days, fifteen minutes of the arithmetic period were given to the part of the class which needed specific remedial help. Five hours were given to the experiment. The following pages suggest the remedial measures used. Several of the remedial units suggested by Messrs. Knight and Buswell in the *Second Yearbook of the National Council of Teachers of Mathematics* were tested.

Sample I—Exercises Stressing Vocabulary

Directions: The words below are often used in your arithmetic work. Following each word you will find four statements, one of which tells correctly the meaning of the word when used in arithmetic work. Put a circle around the number of the explanation which is correct in the case of each word.

A

I. Perimeter
1. Covers a square surface.
2. Measures the total distance around.
3. Is equal to length times width.
4. Is the same as area.

II. Area
1. The length and width of a surface.
2. Amount of land in an acre.
3. Number of square units in a given plane figure.
4. Distance around base of figure.

B

It takes 10 sq. ft. of glass to cover the top of Mr. Bonner's desk. Underline the word given below that most nearly explains meaning of the 10 sq. ft. in the above problem.

Length	Area	Size	Width

C

Complete the sentences below by using the phrases from the accompanying list that make the most complete sense. (These phrases are selected from The Stone Arithmetic—Intermediate, page 1.)

At this rate	his share of it
average per week	total expense
kept an account	share equally
daily sales	have raised $28.88
3/5 of the money	proceeds

1. The milkman ——— ——— ——— of his ——— ——— on Scio St.
2. Oranges are 40¢ a dozen. ——— ——— ——— what will 4 dozen cost?
3. Four boys ——— ——— ——— worth of cabbage. If they ——— ——— the ———, how much will each boy receive?
4. If the ——— ——— of a car for a year was $212 what did that ——— ——— ———?
5. Tom received ——— ——— ——— ——— left by his father. He put ——— ——— ——— ——— into the bank.

D

Reading numbers.

Whole Numbers—Page 7—Stone Arithmetic.
Fractions —Page 312—Stone Arithmetic.
Decimals —Page 204—Stone Arithmetic.

E

Following Directions.

1. Name the topics included under the Sixth Grade, First Semester, in the Table of Contents of The Stone Arithmetic—Intermediate.
2. Turn to page in Sixth Grade, First Semester section, where you can find work in multiplying a fraction by a fraction.
3. Listed under Sixth Grade, First Semester, is a lesson entitled "A Test of Principles Used in Decimals." Turn to it.

Sample II—Exercises Stressing Problem Comprehension

A

Comprehension—Two-Step Problem

Directions: Read each problem carefully. Then read each statement on your slip and write in the empty spaces the letter which stands for the statements which are true, as you understand the problem. Read the problem as often as you need to.

Problem I—Page 1—Stone Arithmetic—Intermediate. (Problems taken when feasible from textbook in the hands of the children, so that familiarity with printed page may be increased):

"During his vacation Walter earned $3.40 per week delivering papers and $1.85 more running errands. How much did he earn in 9 weeks at this rate?"

( ) (a) Walter earned less delivering papers than he did running errands.
( ) (b) Walter worked 9 weeks altogether.
( ) (c) I should divide and multiply to solve this problem.
( ) (d) Walter earned the same amount of money each week.
( ) (e) Walter spent all his time delivering papers.
( ) (f) Walter earned this amount for about one half the year.

B

"We are raising money for a new victrola. Miss Lehrberg's grade collected 280 lb. of papers and Miss Bedford's grade 220 lb. What did the papers bring at ½¢ a pound?"

Price Papers No Yes

If you are asked to find the number of papers Miss Bedford's grade collected, underline the word Yes; if not, underline the word Price.

Sample III—Exercises Stressing What Is Given in the Problem

A

What is Given—Two-Step Problems

Directions: Read each problem carefully. Then read each statement on your slip and write in the empty space every letter which stands for a fact given in the problem. Several of the statements may be true.

Problem III—Page 1—Stone Arithmetic—Intermediate.

"Ralph earned $4.75 per week for 9 wk. during his vacation and spent $27.50 of it for a bicycle. How much of it had he left?

( ) (a) Amount Ralph spent for a bicycle.
( ) (b) Amount Ralph had left.
( ) (c) Amount Ralph earned in one week.
( ) (d) Number of weeks Ralph worked.
( ) (e) Amount Ralph earned in 9 days.
( ) (f) What part of Ralph's money he spent for a bicycle.

B

"It took 15 in. of wire to hang each of four pictures in Mr. Bonner's office. How many feet of wire were required?"

Underline the number given below that tells how many feet of wire the pictures required.

4 5 15 60

Sample IV—Exercises Stressing What is Called for in the Problem

A

What is Called For—Two-Step Problems

Directions: Read each problem carefully. Then read each statement on your slip and write in the empty space the letter which stands for the one statement which tells what is called for by the problem.

"Gertrude Messenger used 2½ yd. of percale for her apron, while Violet, her sister, used only 2¼ yd. If the percale is $.25 a yard, how much must Mrs. Messenger send to Miss Caragher to pay for apron material?"

( ) (a) Amount of material Gertrude required.
( ) (b) Amount of material Violet required.
( ) (c) Total cost of material.
( ) (d) Cost per yard of material.
( ) (e) Total amount used by both girls.

B

"Mr. Brown has $5000 in the bank for his children. When his son Tom reaches the age of twenty-one, he is to have half of it and two daughters are to share the rest equally at the same time. How much does each receive?"

Underline the number given below that tells how many things you are asked to find in the problem.

1 2 3 4

Sample V—Exercises Stressing the Estimation of Answers

A

Probable Answer—Two-Step Problems

Directions: Read each problem carefully. There are five suggested answers (a-e) below the first problem. One of them is the most probable answer of the five. In the empty space write the letter which stands for the most probable answer. Read the problem as often as you need to.

"White's Jewelry Store is about ¾ of a mile from the City Normal School. Highland Park is about 1½ miles. On Monday we visited White's Jewelry Store, walking both ways. On Tuesday we walked to and from Highland Park. What was the total distance that we covered on the two trips?"

( ) (a) About 10 miles.
( ) (b) 4 trips.
( ) (c) 5 miles.
( ) (d) ¾ mile.
( ) (e) Twice as far.

B

"There are 29 children in 6 B 2. 14 went to lunch today, while 5 were being given individual help at the board by Jane. The rest of the class were in their seats doing written problems."

If the number of children doing written problems was greater than the number who went to lunch, underline the word More; if not, underline the word Less.

Lunch More Less Help

Sample VI—Exercises Stressing Choice of Procedure

A

Process Analysis—Two-Step Problems

Directions: Read each problem carefully, but do not try to get the answer. Write the proper letters A, S, M, D, to show which two of the four processes you would use if you were to work the problem.

"Miss Hise used 50 min. for arithmetic every day last week. How many hours last week were given to arithmetic?" ( )

B

This time write the proper signs, +, —, ×, ÷, to show which two of the four processes you would use in solving the following problem.

"Mother's laundry bill was $1.65 the first week in April, $1.84 the second week, $2.01 the third week, and $1.49 the fourth week. What did her laundry bill average for April?" ( )

C

"If you know the number of hours your father worked on Monday, the number of hours he worked on Tuesday, and the price per hour he received for his labor; how will you find his total earnings for the two days?"

Underline the group of signs given below that tells the right things to do:

(+,—), (×,÷), (+,×), (×,×)

SAMPLE VII—EXERCISES STRESSING RELATIONSHIPS IN PROBLEMS

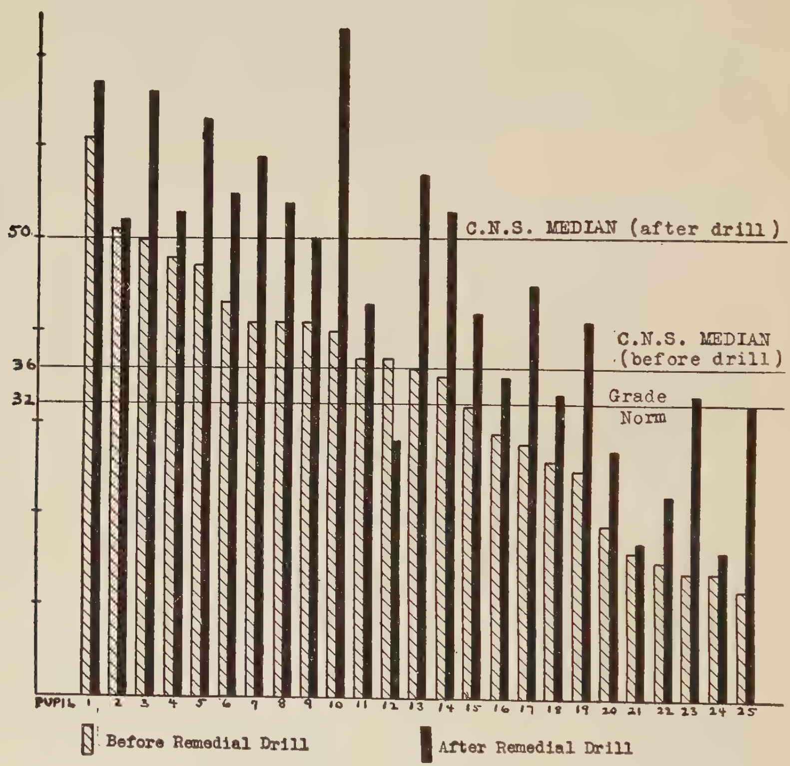
A

"Miss Card drove 3 hr. at the rate of 20 mi. an hour. How far did she drive?"

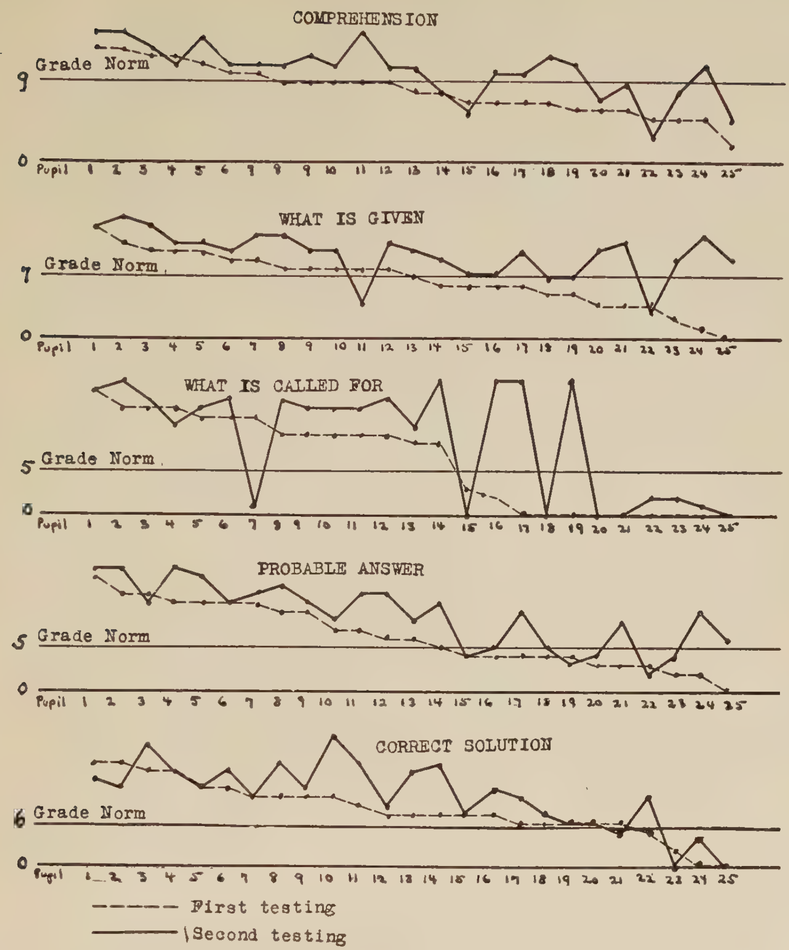
Underline the relationship listed below that is the correct one to use.

(a) Distance = Time + Rate
(b) " = " − "
(c) " = " × "
(d) " = " ÷ "
(e) " = Rate ÷ Time

Results of the Experiment. The accompanying graphs show the results which were obtained in the case of each individual pupil, and also the median scores of the class before and after the experiment.



COMPARISON OF SCORES ON COMPASS DIAGNOSTIC TEST IN PROBLEM ANALYSIS. BEFORE AND AFTER 20 LESSONS IN REMEDIAL DRILL. GRADE 6B2.



Effect of 20 Lessons in Remedial Drill upon Specific Phases of Problem Solving. Grade 6B2. C. N. S. Scores of Individual Pupils.

II. Second Experiment at the City Normal School

Grade 5 A-2 — Critic Teacher—Miss Mildred Seekins
Number of Pupils—14 — Feb. 28 to May 18, 1927

Purpose. To determine the difficulties that pupils had in solving problems, to apply remedial instruction, and to increase, if possible, the ability to solve problems.

Procedure. Tests were given at the beginning of the experiment and at the end of the remedial instruction, to find individual disa-

bilities and to measure the results of remedial instruction. The tests used were the Buckingham Scale for Problems in Arithmetic, Forms 1 and 2, and the Stevenson Problem Analysis Test, Forms 1 and 2.

Remedial Instruction. All the remedial instruction was done in the regular arithmetic period. The instruction was planned to cover a period of ten weeks. Three fifteen-minute periods each week were given to the experiment, making a total of seven and one-half hours. While the work was concentrated on one phase or ability each day, a constant review of the previous phases was kept up, and a final mastery of all the phases was made collectively. Original problems, board problems, mimeographed problems, and book problems were used. The following problem from Buckingham's test may serve as an illustration of the daily procedure:

"Henry gathered 5 qt. of nuts. He sold them at 8¢ a quart and spent the money for oranges at 4¢ apiece. How many oranges did he buy?"

First Lesson. Concentration on: What is required? Example: "How many oranges did Henry buy?"

Second and Third Lessons. Review: What is required. Concentration on: What is given, or the Data. State Data in words as:

Number of quarts of nuts is 5.
Selling price of 1 qt. of nuts is 8¢.
Cost of 1 orange is 4¢.

Fourth, Fifth, Sixth, Seventh Lessons. Review: What is required. What is given. Concentration on: How to solve the problem orally.

$5 \times 8¢$ will give the selling price of nuts.

This result divided by 4¢ will give the number of oranges bought. (Most difficult step; crucial step.)

Eighth, Ninth, Tenth Lessons. Review: What is required. Data. Plan of solution. Concentration on: Oral solution.

$$5 \times 8¢ = 40¢$$
$$40¢ \div 4¢ = 10 \text{ times}$$
$$\therefore 10 \text{ oranges,}$$

or,

$$\frac{5 \times 8¢}{4¢} = 10 \therefore 10 \text{ oranges.}$$

Eleventh, Twelfth, Thirteenth Lessons. Review above. Concentration on: Written solution. Use horizontal or vertical form. In horizontal form the result of each step is labeled and the label appears first to provide for planning before execution. Long calculations may be performed on scratch paper.

A. *Horizontal Form—*

Selling price of nuts, $5 \times 8¢ = 40¢$.
Number of oranges bought, $40¢ \div 4¢ = 10$ $\therefore$ 10 oranges. Or,

Cancellation method if developed—

$$\frac{5 \times \overset{2}{\cancel{8}}¢}{\underset{1}{\cancel{4}}¢} = 10 \quad \therefore 10 \text{ oranges.}$$

B. *Vertical Form*—

$$\begin{array}{r} 8¢ \\ \underline{5} \\ 4¢\overline{)40¢} \\ \overline{10} \end{array} \quad \therefore 10 \text{ oranges.}$$

Fourteenth, Fifteenth, Sixteenth Lessons. Review above. Concentration on: *Checks* on problem-solving. Each problem should be checked either orally or in written form.

1. Check by solving another problem made from the original problem, as— "Henry bought 10 oranges at 4¢ each for the money received for 5 qt. of nuts. How much did he receive for each quart of nuts?"
2. Check by solving the problem by another method if possible. Example: Finding the perimeter.
3. Check by inverse process.
4. Check by going over work another time. Least satisfactory check, as the mind tends to repeat itself. An error made the first time is apt to be repeated.
5. Check by estimating.
   a. Before solving, form an approximate estimate of the answer by substituting the nearest easy numbers for the numbers in the problem. Use this estimate as a check on the answer when the problem is solved.
   b. Put the answer obtained back in its concrete setting to see if the answer is reasonable.

Seventeenth, Eighteenth Lessons. Review above. Search for problems solved in a given way. List page and number of problem as—

Stone	p. 19
M M	5
M S	6
S A	8
S D	10

Nineteenth, Twentieth Lessons. Review above. Children originate problems to fit given conditions. Data given. Children build around these the detailed statements.

Twenty-first, Twenty-second Lessons. Children originate problems to solve in a prescribed way, as—

"Give a problem in A D, M S, D A, etc."

Children originate problems, and other children give the processes.

Twenty-third, Twenty-fourth Lessons. Review above. Concentration on: Problems without numbers. Children read and originate problems without numbers and tell the processes.

INDIVIDUAL AND CLASS RECORDS

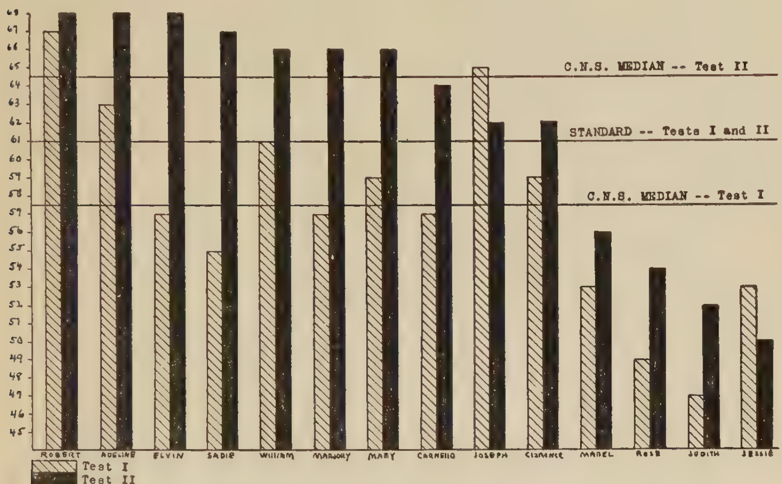
Name	Buckingham Problem Test			Stevenson Problem Analysis Test														
				A			B			C			D			Total		
	Correct		Gain	Correct		Gain	Correct		Gain	Correct		Gain	Correct		Gain	Correct		Gain
	1	2		1	2		1	2		1	2		1	2		1	2	
Robert	67	68	1	4	5	1	4	6	2	6	5	—1	6	6	0	20	22	2
Joseph	65	62	—3	2	4	2	6	6	0	6	6	0	6	6	0	20	22	2
Adeline	63	68	5	4	6	2	6	6	0	5	5	0	6	6	0	21	23	2
William	61	66	5	5	6	1	2	6	4	6	5	—1	6	6	0	19	23	4
Mary	59	66	7	4	5	1	6	6	0	4	6	2	5	6	1	19	23	4
Clara	59	62	3	5	5	0	5	6	1	6	6	0	6	6	0	22	23	1
Elvin	57	68	11	5	6	1	1	0	—1	3	4	1	5	6	1	14	16	2
Marjory	57	66	9	3	5	2	5	4	—1	6	4	—2	6	4	—2	20	17	—3
Clarice	57	64	7	2	5	3	6	2	—4	5	3	—2	4	6	2	17	16	—1
Sadie	55	66	11	5	6	1	3	4	1	4	4	0	4	6	2	16	20	4
Mabel	53	56	3	4	5	1	3	3	0	2	3	1	2	6	4	11	17	6
Jerry	53	50	—3	5	5	0	3	4	1	4	3	—1	5	6	1	17	18	1
Rose	49	54	5	4	2	—2	4	2	—2	1	3	2	1	2	1	10	9	—1
Judith	47	52	5	3	4	1	5	3	—2	3	4	1	3	6	3	14	17	3
Total	802	868	66	55	69	14	59	58	—1	61	61	0	65	78	13	240	266	26
Average	57.29	62	4.71	3.93	4.93	1	4.21	4.14	—.07	4.36	4.36	0	4.64	5.57	.93	17.14	19	1.86

Buckingham Problem Test			Stevenson Problem Analysis Test		
	Form 1	Form 2		Form 1	Form 2
Standard Median	61.0	61.0	Standard Score	16	14.7
Class Median	57.5	64.5	Class Score	18	19.0
Above or Below	—3.5	3.5	Above or Below	2	4.3

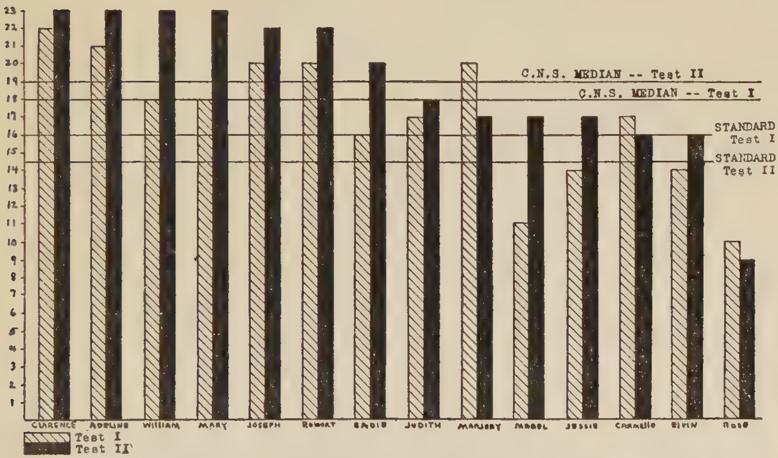
Twenty-fifth, Twenty-sixth, Twenty-seventh, Twenty-eighth Lessons. Review and clinch all phases, steps, and abilities of problem-solving collectively.

Twenty-ninth, Thirtieth Lessons. Final tests given.

Results of the Second Experiment. An examination of the accompanying medians, scores, and graphs for first and second trials on the problem tests shows the individual and the class improvement or lack of improvement for the experiment.



COMPARISON OF SCORES ON BUCKINGHAM PROBLEM SCALE, GRADE 5A2. FEB. 28-MAY 18, 1927.



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A MATHEMATICAL ATMOSPHERE

BY OLIVE A. KEE
Teachers College, Boston, Mass.

Atmosphere of the Classroom. Should the mathematics classroom breathe mathematics? Is it possible to develop an arrangement of materials which will make the appearance of the mathematics classroom as distinctive as that of a room where science or history is taught?

It must be admitted that in the majority of cases the visible evidences that a classroom is devoted to mathematics are few and uninteresting. Possibly a set of graphs, which have been cut from newspapers and magazines, and which may or may not mean anything to the students who casually glance at them, is displayed above the blackboard and above eye-level. Or, much better, models of the various geometric forms may be displayed in the cabinet. They may have been constructed by the students, in which case they surely have some meaning. But, with these exceptions, one is likely to find little of significance.

The writer was considering such matters a few years ago, and trying to think of ways and means of making the demonstration classroom in junior high school mathematics vitally attractive. At that time the council of the Association of Teachers of Mathematics in New England was considering such questions as: Who is to convince the casual public of the worth and power of modern mathematics? How is the teacher to stimulate controversy which will finally lead to the understanding and appreciation of the values of her subject? In other words, how can we sell our subject to the "man in the street"? Professor Copeland, then president of the Association, suggested that one of the members make an experiment with posters advertising mathematics. The writer undertook to try out this suggestion, in the hope that it would afford a solution not only to the problem propounded by the Association, but also to the question which was in her own mind.

Mathematics Posters. Posters were, of course, rather generally

used in the English and hygiene departments of our own and other institutions at that time. No one questioned the fact that a good poster advising us to drink more milk has a more definite appeal than the bare spoken or written message. Posters in the mathematics classroom, however, were an innovation.

The matter was first taken up with those of our students who were preparing to teach in the elementary schools. Considering their interest in the work of the lower grades, the writer saw various possibilities in enlisting them in the poster-making enterprise. Posters of the following sorts were suggested:

1. Those which would show the uses of mathematics in society.
2. Those which carried in their slogan a positive and definite suggestion for the pupil in connection with certain classroom tasks.
3. Those intended to stimulate younger pupils to greater endeavor.
4. Those showing geometric forms in nature and art.

Worth of Mathematics. In order to set the project moving, the students, who had previously reached conclusions as to the objectives in the teaching of mathematics, engaged in a general discussion of the worth of mathematics in the trades, industries, and sciences. It is worth noting, as an aside, that the resulting posters, although exaggerated in many respects, never once claimed broadly that "mathematics trains the mind." Whether the omission was due to fear of putting this statement into lasting form, or whether no picture could be found to illustrate the point, cannot be determined. It may, however, be assumed that the students were honestly trying to show, as Dean Holmes puts it, "what man can do with mathematics; not what mathematics can do to man."

Uses of Mathematics. While the students were thoroughly imbued with the idea of present use or interest as an objective in the teaching of any topic, they recognized the fact that interest in a trade in which arithmetic plays a part has been known to stir up enthusiasm for the necessary topic. A practical use of this fact was made in the preparation of posters falling in the first category. After all, young pupils trust their teachers, and a sixth grade boy may quite conceivably work on a process because his teacher knows it to be valuable in connection with some interesting trade or industry. The teacher must here, of course, have an adequate background, and know how to complete the task successfully.

For example, in grade eight, a poster may be shown advertising the use of the formula in computing compound interest. The pupils are arriving at the result by computing amounts period by period. The teacher shows them how she obtains results in a much shorter time by her formula. Whether they can use the formula or not is inconsequential at this time. The poster showing the bank and its sign is left in sight for a day or two. Will it not perhaps inspire a future Rothschild with the idea that banking depends at least in part on a phase of mathematics which is something unused by him? Again, a good poster displaying the graduated test-tube, beaker, and scales, and carrying the slogan "Chemistry depends on Mathematics" will probably start a controversy about our subject, if not an appreciation of its value.

Use of Mathematics in the Classroom. Suggestions for a second type of poster were also given. Posters with a warning about checking and about the making of an estimate were agreed upon as a valuable possession for the young teacher who was preparing for a hard day in the classroom. Other phases of process-work in arithmetic and algebra were mentioned as having possibilities. As later illustrations will show, these hints were followed quite generally but to the inhibition, it seemed, of a variety of original ideas.

One illustration of this sort was the well-known check-mark ($\surd$) with the slogan "Stop! Have you checked?" The position of the check-mark in mid-air, a bit like the sword of Damocles, was apparently intended to intimidate the careless. This poster would be useful even in the fourth grade. Its value, like that of all the others, is brought out by displaying it at the psychological moment and only then. Otherwise it will become like the spot in our fence which, needing repair, at first sticks out like a sore thumb and later becomes an accepted part of the landscape.

Another of this type which was more suited to the seventh grade had as its slogan, "Members of an equation are like twins. Treat them alike." The picture was that of a girl between two twins presenting like material to them. The students felt that this would be effective in the early stages of work on the equation. Another striking poster carried the slogan "A Check is Valuable."

Use of Incentives. One runs the risk of criticism for suggesting the third type. A single illustration will suffice. This poster shows a boy climbing a ladder with rungs labeled from 10% to 100%. "Can you reach the top in arithmetic?" is the slogan. Of course,

we know that some pupils would necessarily answer this question (if at all) with "No" unless more drastic measures were used—"drastic" signifying nothing worse for the pupil than individual work with flash-cards or survey sheets.

Some amusing posters in this third class were "Can you make your letter in arithmetic?" under a picture of a slightly-built boy gazing at a much augmented reflection of himself, the letter "A" appearing on his sweater. Another was "How many medals can you earn in arithmetic?" Here again the undecorated boy sees his reflected chest covered with medals. However pleasing and amusing these are, little change would take place in the schoolroom situation without a great deal of stimulus in the way of arithmetic contests.

To be sure, older people usually gasp with pleased surprise when they see such posters on the wall of the classroom. Their reaction, however, is no criterion. Peter Pan and also some of our best stories about children give more pleasure to adults than to the boys and girls.

Geometric Forms in Nature and Art. The fourth variety of poster did not occasion so much discussion. The head of the art department in our school constantly emphasizes the idea of mathematical law underlying art. As illustrations of geometry in nature the students furnished pictures of snow-crystals, and of fruit cut to show the symmetrical arrangement of the seed-pods. One student contributed a beautiful picture of a church window, with the slogan "Geometry in the quiet of the church." This poster made a strong appeal to many.

Further Uses of Posters. Thus far only one part of the experiment has been described,—that in which our own students at Teachers College engaged. The group preparing to teach in the junior or senior high school had the same instructions as the group working in the elementary grades. In general they chose to show the uses of mathematics as applied in various industries or sciences. "The plan made this house. Mathematics made the plan" was effective with its picture of a house and the ground plan. One fine exception was an interesting and beautiful poster illustrating the addition of negative numbers.

Next a group of teachers who were taking an extension course in junior high school mathematics were asked to try out the project. All were urged to bring in one poster which they considered worthy

of display in the classroom. Those who were teaching in grades above sixth were also urged to have their pupils make posters. *Geometry for Junior High Schools*, by Betz, was mentioned as full of suggestions. Although the teachers were reluctant to begin work upon the assignment, the results were most interesting and stimulating. It might not be wise, however, to quote these slogans to an educational department already branding all specialists in mathematics as over-enthusiastic about the possibilities of the subject. For example, the picture of a broadcasting station carried the slogan, "Stand by! The world is calling for mathematics." And yet the world has been doing so since time immemorial, and the call is still heard. Even if this poster did no more than start controversy, it would have served its end, provided that someone knew enough not to let the discussion die at its inception.

Other offerings started discussions immediately, among them being pictures of bridges bearing the statement that these could not have been built without the use of mathematics. The whole subject of results found only by experimentation versus results predictable by mathematics came up for debate. Dewey's *Psychology of Number* was used as an authority by one side.

A poster picturing Niagara Falls with the slogan, "Mathematics has harnessed the elements for the service of man," was admired and not challenged, as was also one with the "Cathedral of Commerce" (the Woolworth Building) and the slogan, "Mathematics made this possible." Another favorite was "Mathematics—the master key." Here was displayed a massive key, flanked by doorways labeled Art, Science, Astronomy, Agriculture, Commerce, and Manufacturing.

The most interesting feature of the experiment was the part played by the young pupils in the junior high schools. Many of the teachers had results from their classes which were interesting if not beautiful. The best collection was furnished by Miss Susan Hosmer of Washington Irving Junior High School. The posters were not made during the mathematics period although the stimulus was given at that time. The pupils in this school had the advantage, it seemed, of good materials and a good foundation in art and printing. The results were original even if the slogans were at times debatable. One which no one could question was a "life-size" automobile number plate, "Mass. 448,914," with the canny comment, "One needs a slight knowledge of mathematics to read

this." Another which escaped with little criticism had a good picture of the Capitol at Washington with "King Math helped to make this beautiful structure."

Since the class had a majority of boys, boy-interests figured almost entirely. The arts and sciences of war or defense were unduly stressed, as might be expected. Examples of this are:

"The men who fired the shot could not see the target but mathematics helped them to hit it," accompanied by a target.

"Mathematics rules the undersea," with a cross-section of a submarine.

"Aviation—$\frac{9}{10}$ mathematics," with a dirigible.

"The lives of two men depend upon mathematics"; picture of airplane and two fliers.

"Mathematics is used to direct naval maneuvers"; naval officers working over a map.

"Mathematics helps these men to defend the nation"; Army, Navy, and Marines.

Less warlike are such posters as these:

"The car that Math. built"; a well known automobile.

"Astronomers use mathematics in calculating distances"; picture of planets each with a train apparently headed toward it.

"Mathematics solves the problem of traveling"; a locomotive.

"Mathematics helps us to make accurate maps"; a map.

Far-fetched? In some cases. Questionable? True. Yet the boy of the junior high school age is ready to challenge and to be challenged. If he tries to meet challenge with accurate information, and if his parents take sides, then the casual public is aroused if not convinced.

Efforts should, of course, be made to have posters illustrating the use women make of mathematics. One of our most beautiful posters was that of "Geometry in the home,"—a girl making a lampshade with hexagonal base. An amusing one was "Know your proportions"—a luncheon scene with the small son complaining of too much salt. Another had "Math. in the making, Math. in the baking" as its slogan, at the risk of having the enemies of mathematics refuse the luscious cake.

In the same collection was one (by a girl) which amused some and annoyed others. This was a picture of a piano with the words "Harmony is attained through Mathematics." As one young col-

lege student expressed it, "I love music and always did, just as I always hated mathematics. But if anyone had suggested to me that mathematics was necessary in the study of music, I should have given up music." This is the other side of the picture. From our experiment we are persuaded that stressing the many uses of mathematics may, we do not say will, help us to interest students in it. In any event, there is much evidence to show that students took a livelier interest in mathematics because of the poster project. One of these youngsters in the junior high school said enthusiastically to her teacher, "Why, there's mathematics in almost everything."

The Real Purpose of Posters. In justification of some of the posters it may be said that although they do not encourage the pupils to be producers in the mathematical field, they awaken interest and stimulate discussion, thus making the pupils more intelligent consumers of mathematics. It really seems as if posters made by the pupils may interest even the casual public in mathematics.

Our own students are still trying to make posters. The writer is positive that they are pleased when they see fresh ones on display in the classroom. The present class showed surprise at the first exhibition. More than one student has frankly admitted that mathematics had never before meant anything to him except problem solving.

It must be admitted that thus far the efforts of the students have not as a whole received the approval of the art department. To win that approval is the next step. The students have definite directions furnished by the art department as to the making of good posters. When worthwhile ideas are carried out effectively, our collection of posters may be used to convince the public of the values of mathematics as well as to create a mathematical and artistic atmosphere.

Graphs. As for our mathematical atmosphere, posters are only incidental, albeit important. They are not the sum total; graphs also have their place. A plea is made, however, that the collection of graphs be not allowed to degenerate into an array of meaningless clippings. If the pupil contributing the graph can also interpret it to the class, then it may be given a place on the wall or bulletin board. My own students have showered me with graphs meaningless to them. They may, indeed, ask their teacher to help them to interpret these, and thus become real consumers.

Graphs cut from papers suggest graphs made by the pupils. About a year ago, a collection of this sort in many schools would have seemed impressive. Even now, the mathematical atmosphere can be at times improved by having these graphs relate to the pupils' own activities. Diaries kept over a period of two weeks or so to give an accurate budget of time furnish good material for a circular graph which not only is mathematically correct but brings all kinds of good concomitants. For older pupils an allowance budget accompanied by the circular graph is a worthwhile exhibit for the mathematics classroom.

Correlation with other subjects leads to graphs which make the mathematics classroom not merely a place where answers are obtained and where the value of x is found. Our students have displayed graphs on the cleaning of teeth before and after a health talk—a vital subject for them. They have shown by means of graphs the different nationalities which may be represented in a real classroom.

Other circular graphs had to do with these subjects: Expenses incurred by my home. How mother spends Saturday. How I spend my day. The number of pupils in each year at our school. How modern college girls spend their time. A menu for each meal (by calories). The ideal human diet. The initial cost of a small poultry yard. Weather in New England. An evening's entertainment by radio.

In addition, there are always graphs illustrating geographical facts. These graphs, well drawn, with good printing and water-coloring, are quite as interesting as those found in the texts. They seem, when well arranged, to contribute much to the mathematical atmosphere of the room.

Newspaper Clippings. All clippings on the bulletin board should be meaningful. Otherwise they are like reeds shaken by the wind. A room uncluttered by them would be preferable. It is surprising how many good mathematical clippings one can find in the maligned newspaper. Hard upon our discussion of the metric system comes an article predicting its adoption in the near future. It goes without saying that some one will attempt the trisection of an angle every now and then. Sometimes a good collection of jokes about mathematics and its possibilities will add its bit to the atmosphere. In all cases the clippings should be commented upon before being posted.

Reading Table. Surely there should be a table in the corner with supplementary reading material in mathematics as in history or literature. Here should be found Smith's *Number Stories of Long Ago,* Weeks' *Boy's Own Arithmetic,* the geometry mentioned before, and even mere texts other than the ones used in class.

Other Equipment. In the proper place, beside the time-honored models, should be the carpenter's level, the decimalized tape, a good slide-rule in addition to the ones made by the pupils, and other tools. Some of these things are likely to make their appearance from time to time and, having appeared, they should be made available.

Summary. In conclusion it may be reiterated that other features than the work on the blackboard should suggest mathematics. Is the atmosphere such that all may realize that the teacher loves her subject enough to try to win others to its practice and delights? May even those who dislike the mechanics of mathematics be led to admit freely that the subject lives? If our classrooms breathe mathematics before and after, as well as during class periods, we may feel that we have taken a step in the right direction.